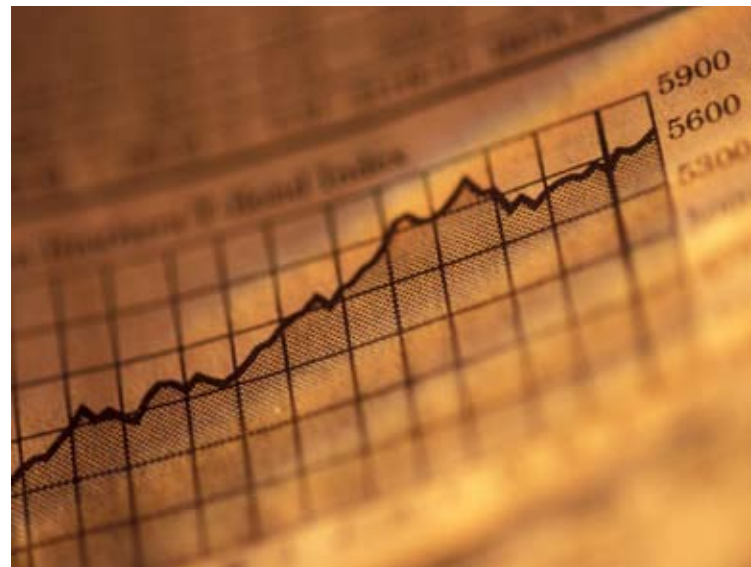


Linear Programming

Linear Programming



Prepared by David K. J. MTETWA



African Virtual university
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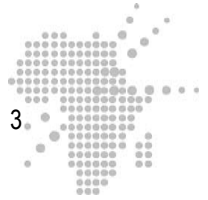
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I. Linear Programming

by David K J Mtetwa

II. Prerequisite Courses Or Knowledge

The Prerequisite courses are Basic Mathematics and Linear Algebra, which are offered in this degree program. A knowledge of linear independence, basis, matrix operation, inverses, inequalities, vector spaces, convex sets, and graph plotting is essential. These concepts and skills are generally covered in the pre-requisite course (or equivalents) mentioned above and constitute important background knowledge required to undertake this module. A basic understanding of these and related concepts, and reasonable competence in related manipulative skills (such as matrix and graphical representations and associated algebraic manipulations), are essential background knowledge for this module. Familiarity with these basic concepts and skills, which are assumed in this module, must be secured first before proceeding with the module.

III. Time

The recommended total time for this module is at least 120 study hours, with Unit 1 taking 40 hours [20 hours for each of the 2 Activities], and Unit 2 taking 80 hours [20 hours for the first Activity, 34 hours for the second Activity, and 20 hours for the third Activity], and the remaining 6 hours to be allocated for the pre-assessment (2 hours) and summative (4 hours) evaluation activities.

IV. Material

Students should have access to the core readings specified later. Also, they will need a computer to gain full access to the core readings. Additionally, students should be able to install the computer softwares wxMaxima and Graph and use them to practice algebraic concepts. These should be regarded as learning materials to facilitate easier accessing and processing of the core concepts and skills that constitute the course. The following are materials necessary to engage with the module meaningfully and, hopefully, complete it successfully: The student's edition of the module (print form); a computer with effective internet connectivity and MicroSoft Office 2003 and above; a scientific or programmable calculator; graph plotting materials; CDs with materials downloaded from sites recommended in the module; CDs with mathematical software such as MathType or WinShell, Graph, wxMaxima, and at least one linear programming software that is free-downloadable, recommended readings from texts identified in the module. [The recommended readings can also be in print form].

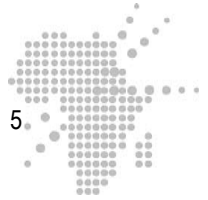


V. Module Rationale

The importance of linear programming derives in part from its many applications and in part from the existence of good general-purpose techniques for finding optimal solutions. Linear programming is useful for guiding quantitative-related decision making in business, industrial engineering enterprises, and to a lesser extent activities within the social and life sciences. The linear programming skills will help teachers in some aspects of their own personal life management activities and in their professional practice.

This module acts as a smooth and non-intimidating entry into the mathematical worlds of dynamic linear programming, networks, and operations research for the learner who will develop some interest in majoring those fields. Also it:

- (a) is important in and of itself as a degree level mathematics course because it introduces to the mathematics student new mathematical content with a distinctive style of mathematical thinking;
- (b) beautifully integrates theoretical concepts with their practical applications – both of a mathematical and everyday - life in nature;
- (c) is necessary for the prospective teacher of science and mathematics because modern day youth and school students are now pre-disposed to a range of career interests, many of which would be facilitated by a preparation that involves dealing with linear programming and optimisation that are covered in this module.



VI. Content

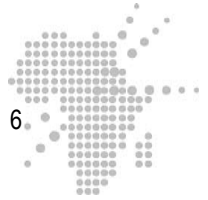
6.1 Overview

Overview: Prose description

This module introduces the learner to a particular mathematical approach to analysing real life activity that focuses on making specific decisions in constrained situations. The approach, called linear programming, is presented here with an emphasis on appreciation of the style of thinking and interpretation of mathematical statements generated, rather than on computational competency per se, which is left to appropriate and readily available ICT software package routines.

The module begins with Unit One that consists of 2 main Activities. Activity 1, formulation of a linear programming problem, is on a mathematical description of the problematic situation under consideration, and Activity 2, the geometrical approach considers a visual description of a plausible solution to the problem situation. Unit 1 therefore should move the learner towards an appreciation of real-life activity situations that can be modelled as linear programming problems.

With 3 main activities, Unit 2 considers computational algorithms for finding plausible optimal solutions to the linear programming problem situations of the type formulated in Unit 1. Activity 3 examines conditions for optimality of a solution, which is really about recognising when one is moving towards and arrives at a candidate and best solution. Activity 4 discusses the centre piece of computational algebraic methods of attack, the famed Simplex algorithm. This module focuses on the logic of the algorithm and the useful associated qualitative properties of duality, degeneracy, and efficiency. The final Activity touches on the problem of stability of obtained optimal solutions in relation to variations in specific input or output factors in the constraints and objective functions. This so called post optimality or sensitivity analysis is presented here only at the level of appreciation of the analytic strategies employed.



Flow of Learning

Unit 1

Identifying, describing, understanding, and appreciating the general linear programming problem situation, and plausible solutions for it.



Unit 2

Computational strategies for seeking solutions of linear programming problems, recognizing potential and best solutions, and efficiency considerations.

6.2 Content Outline

Unit 1: The linear programming problem

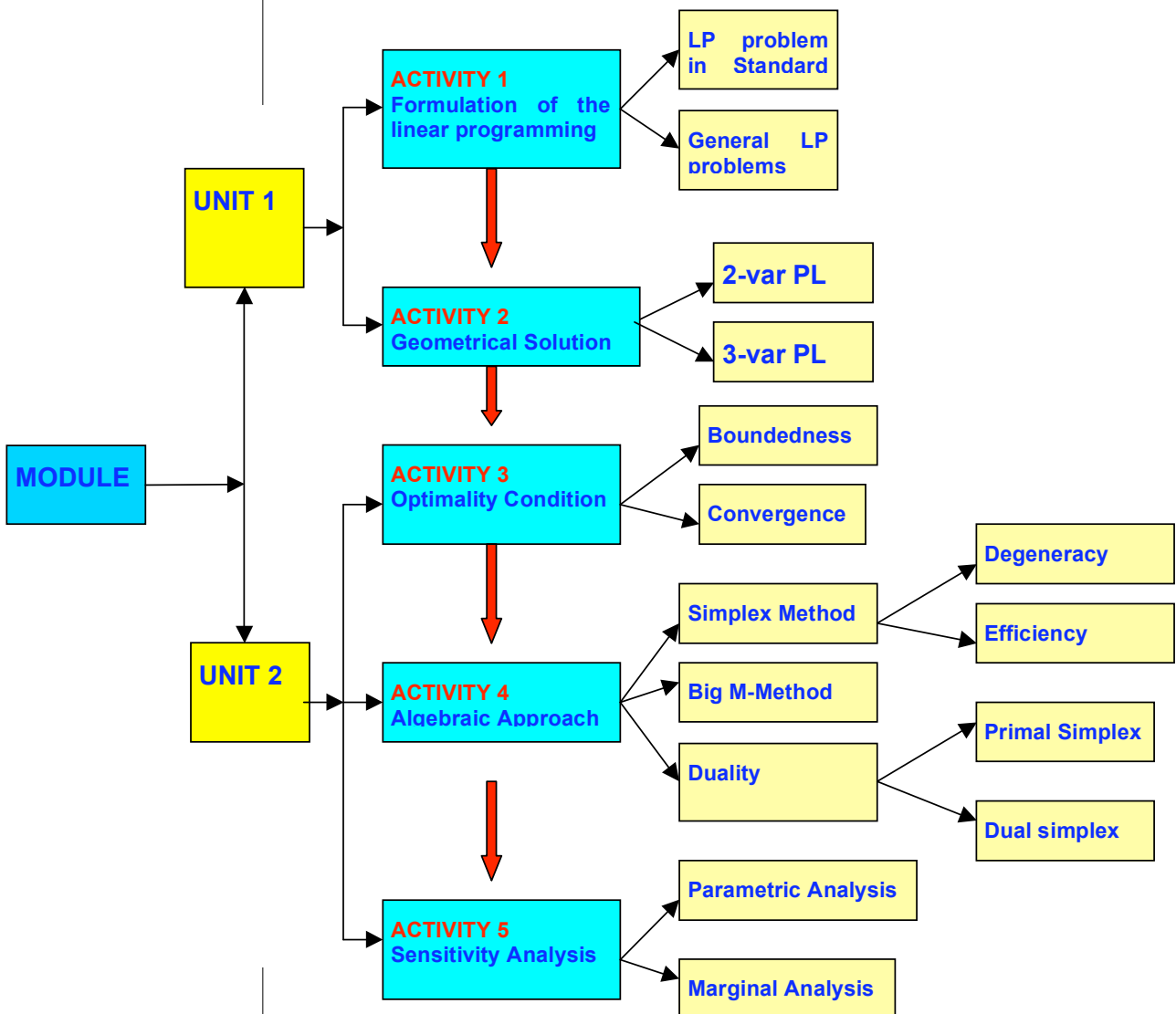
- Formulation of a linear programming problem
 - The general linear programming problem
 - The standardized linear programming problem
- Geometrical interpretation of a solution of a linear programming problem
 - Two dimensions
 - More than 2 dimensions

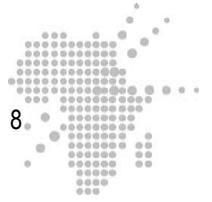
Unit 2: Computational algorithms

- Searching for and recognising a potential solution: optimality conditions for the objective function of a linear programming problem
 - Boundedness
 - Convergence
- Algebraic interpretation of the solution to a linear programming problem
 - The Big M algorithm
 - The Simplex algorithm
 - degeneracy
 - efficiency
 - Notion of duality
 - the primal simplex
 - the dual simplex
- Stability considerations for a solution: sensitivity analysis
 - Marginal analysis
 - Parametric analysis



6.3 Graphic Organiser





VII. General Objective(s)

Upon completion of this module students should:

- a) have a general appreciation of the types of problems which are amenable to analysis using linear programming
- b) be able to formulate linear programming problems and solve them using geometrical and linear algebraic techniques.
- c) be able to use mathematical software packages to solve linear programming problems
- d) be able to discuss some theoretical notions of linear algebra and geometry with concrete/practical contexts.
- e) have developed some familiarity with the language of operations research
- f) have developed a sense of algorithmic thinking



VIII. Specific Learning Objectives (Instructional Objectives)

You should be able to:

1. Identify and effectively model suitable problems with linear programming.
2. Apply knowledge of inequalities to solving optimisation related problems.

You should secure your knowledge of school mathematics in:

1. Linear functions and simultaneous equations.

You should exploit ICT opportunities in:

1. Using graph drawing software to investigate linear functions.
2. Using Computer Algebra Systems (CAS) software to solve linear systems.

Unit	Learning objective(s)
Problem Formulation	<ul style="list-style-type: none"> o The learner should be able to identify/recognize an optimization situation in real life decision making activity. o The learner should be able to produce a mathematical model using appropriate language.
Problem Formulation	<ul style="list-style-type: none"> o The learner should be able to describe, explain, and apply optimality conditions to specific linear programming problems. o The learner should be able to describe the underlying logic of the Simplex algorithm. o The learner should be able to relate the algebraic solution with the geometric solution. o The learner should be able to perform the Simplex algorithm on specific problem situations with an appropriate linear programming software and interpret the resulting solution. o The learner should be able to explain the meaning of duality and describe its role in the search for solutions of linear programming problems. o The learner should be able to describe the purposes of carrying out a sensitivity analysis for a given linear programming solution. o The learner should be able to describe a procedure for carrying out a sensitivity analysis to a given optimal solution.



IX. Teaching and Learning Activities

Pre-assessment

Title of pre-assessment: Basic Algebraic Ideas Test

Rationale: To check learner's familiarity with some concepts assumed in the module

Questions

1. Which of the following is linear:

- (a) $ax^2+by = c$ (b) $ax - by = c$ (c) $ax + by^2 = c$ (d) $a\cos x + by = c$

2. Which of the following does **not typically** denote a vector:

- (a) -5 (b) (1, 2, 3) (c) \underline{A} (d) $\begin{bmatrix} 4 \\ 8 \\ -3 \end{bmatrix}$

3. The matrix $\begin{bmatrix} 2 & 4 & 2 & 2 & 0 \\ 1 & 1 & 6 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is:

- (a) 3×5 (b) 2×5 (c) 5×3 (d) 5×2

4. A singular matrix is one that:

- (a) is single (b) is invertable (c) is non-invertable (d) has determinant 1?

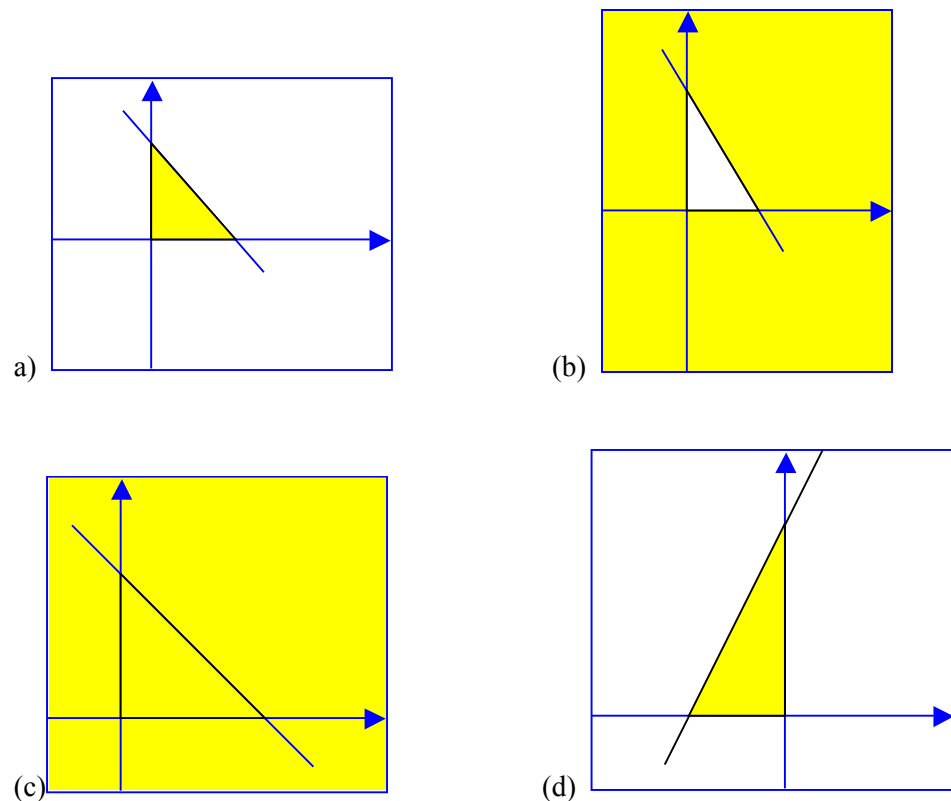
5. The matrix $\begin{bmatrix} 2 & 4 & 2 & 2 & 0 \\ 1 & 1 & 6 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \end{bmatrix}$ has rank:

- (a) 0 (b) 1 (c) 2 (d) 3



6. Consider the linear equation $x + y \leq 1$. Which of the graphs satisfy this inequality?

N.B. The wanted region is shaded.



7. If $ax + by \leq c$ for some numbers a, b, c then $ax + by + d = c$ for some positive number d .

(a) TRUE (b) FALSE

8. Which of the following is **not** directly associated with the Gaussian elimination method?

(a) Reducing $n \times n$ matrix to echelon form
 (b) Determining the consistency of a system
 (c) Finding lower or upper triangular matrix
 (d) Using elementary operations to reduce a system of equations



9. What is the transpose of matrix A?

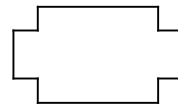
$$A = \begin{bmatrix} 4 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & -1 & 5 \end{bmatrix}$$

$$(a) \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 5 \\ 4 & 1 & 3 \end{bmatrix} \quad (b) \begin{bmatrix} 4 & 1 & 2 \\ 1 & 0 & -1 \\ 3 & 2 & 5 \end{bmatrix} \quad (c) \begin{bmatrix} 4 & 5 & 3 \\ 2 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \quad (d) \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 1 \end{bmatrix}$$

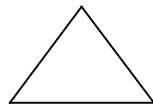
10. Which of the following shows a convex set?



(a)



(b)



(c)



(d)

N.B. An object is convex if for every pair of points within the object, every point on the straight line segment that joins them is also within the object.

11. The sequence $\{1/n\}$ is bounded below by:

- (a) 1 (b) $\frac{1}{2}$ (c) 0 (d) n where n is very large?

12. At an optimal value of a function:

- (a) The function attains a local maximum value only.
 (b) The function attains a local minimum value only.
 (c) The function attains a maximum or minimum value.
 (d) None of the above statements is true?

13. The following is an example of a discontinuous function.

- (a) $y = |x|$ (b) $y = 1/x$ (c) $y = 3$ (d) $y = e^{-x}$



14. A basis of a vector space
- (a) Has linearly dependent vectors.
 - (b) Is a set of only unit vectors.
 - (c) Has basis vectors that span the whole vector space.
 - (d) Can include the zero vector.
15. Which of these sequences does **not** converge?
- (a) $\{(\sin(n))/n\}$ (b) $\{\ln(n)\}$ (c) $\{1/n\}$ (d) $\{(n+1)/n\}$
16. A function is said to be bounded if:
- (a) It is defined in open interval.
 - (b) It is limited above and below only.
 - (c) It is limited above only.
 - (d) It is limited below only?
17. The determinant of the following matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ is:
- (a) 11 (b) 24 (c) 21 (d) 18
18. Maximising profit generation can be equivalent to minimising production costs:
- (a) TRUE (b) FALSE?
19. A vector has:
- (a) magnitude only.
 - (b) direction only.
 - (c) magnitude and direction only.
 - (d) magnitude, direction and field only?
20. A Subspace of vector space:
- (a) Is also a vector space.
 - (b) Is not a vector space.
 - (c) Is not a linear space.
 - (d) Is half of a vector space?



Answer Key

1. b (a and c are parabolic functions, d has a periodic function $\cos x$)
2. a (a has magnitude but no direction, c typically denotes a matrix or vector, b and d are vectors)
3. a (the matrix has 3 rows and 5 columns and so it's a 3 by 5 matrix)
4. c (a singular matrix is a matrix with determinant zero and so it non-invertable)
5. d (the rank of a matrix is the number of non-zero rows and here they are three)
6. a (the graph in d gives the equation $x - y \leq 1$, b has shaded the unwanted region and c has shaded the whole x-y plane)
7. a (for equality to hold you add some positive number on the left hand side)
8. c (you use the leading entry in each row to eliminate the non-zero coefficients below it, thus creating an upper triangular matrix)
9. b (for transpose the first column becomes first row, second column becomes second row and third column becomes third row so answer is b)
10. c (c is the only shape in which when two points in the region are joined by a line, all the points on the line are also in the shape)
11. c (the sequence is a decreasing sequence which approaches zero)
12. c (the optimal point is the maximum or minimum point)
13. b (for b the function is continuous everywhere else except at zero hence discontinuous)
14. c (for a vector to be called a basis it should be linearly independent and should span or generate the whole vector space)
15. b (a and c converge to zero, d converges to 1)
16. b (a function is bounded if it is bounded above and below)
17. b (just multiply the terms in the diagonal) .
18. a (if you minimize production costs then you will maximize profit)
19. c (a vector quantity has size and direction e.g. velocity, acceleration)
20. a (a subspace of a vector space is also a linear or vector space since it satisfies the three main properties of a vector space`)



Pedagogical Comment for Learners

If you get below 50% in this pre-assessment it might indicate that you have forgotten some facts of linear algebra. These might include things like simple definitions, properties and computational procedures for such objects like matrices, vectors, sets, linear systems of equations, and real numbers. In that case you are encouraged to browse through the module on basic mathematical ideas and linear algebra before proceeding. If you get more than 50% you are also encouraged to review the same module as needed while proceeding with this module. Some of these basic ideas will surface as assumed knowledge in one form or another in this module.



X. Learning Activities

[*Note:* all key concepts are defined in the glossary, which is given in section 11 below. Where a definition is encountered in the learning activities, reference is made to the glossary for its articulation, while the Learning Activities concentrate on developing the concept or skill that is carried by the definition. This is a device to minimise repetition.]

UNIT 1: PROBLEM FORMULATION

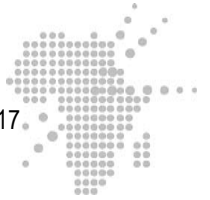
Learning Activity # 1: Formulation of a Linear Programming Problem

Specific learning objectives

- The learner should be able to identify/recognize an optimization situation.
- The learner should be able to produce a mathematical model using appropriate language.
- The learner should be able distinguish and relate between the Standard and the General form.
- The learner should be able represent the mathematical model geometrically.
- The learner should be able identify feasible regions, vertices, and convexes.
- The learner should understand the concepts of systems of linear equations, constraint, feasible solution and feasible region.
- The learner should be able to interpret a real life problem and transform it into a linear programming problem.
- The learner should be able to check or verify the feasible solution.
- The learner should be able to express the system of linear equations graphically, i.e. the learner should understand the geometry of the linear programming model.
- The learner should be able to resolve the linear programming problem by the geometrical approach.

Summary of the learning activity

The subject of linear programming has its roots in the study of linear inequalities. In this unit we give an introduction to linear programming starting with a simple real life problem which the learner can easily relate with. In linear programming the objective is to maximize or minimize some linear functions of quantities called decision variables. This can be done algebraically or geometrically. However here we



resolve the linear programming problem geometrically. With specific examples you are going to see how problems in business, physical, chemical and biological sciences, engineering, architecture, economics, agriculture and management are formulated. Learning will involve a number of activities such as reading about linear programming, optimization, operation research and the nature of the objective functions and constraints. The learning activities could also involve extensive use of mathematical software packages to solve the linear programming problems.

Key Terms (refer to glossary)

Linear programming
Problem formulation
Objective function
Optimal solution
Constraints
Feasible solution
Basic solution
Basis
Basic variables
Dictionary
Nonbasic variables
Slack variable
Surplus variable
Artificial variable
Unbounded solution

List of required readings

Linear Programming: Foundations and Extensions by Robert J. Vanderbei

www.princeton.edu/~rvdb/LPbook/onlinebook.pdf

Last visited: 15-02-07

Lecture notes on Optimization by Pravin Varaiya

http://robotics.eecs.berkeley.edu/~varaiya/papers_ps.dir/NOO.pdf

Last visited: 10-02-07

A gentle approach to linear programming: chapters 1-7

<http://www.sce.carleton.ca/faculty/chinneck/po/Chapter1.pdf>

Last visited: 08-03-07



List of relevant useful links

Linear Programming Formulation

<http://people.brunel.ac.uk/~mastjjb/jeb/or/lpmore.html>

Last visited: 15-02-07

Linear Programming Formulation

<http://home.ubalt.edu/ntsbarsh/opre640a/partVIII.htm>

Last visited: 15-02-07

An Introduction to Linear Programming and the Simplex Algorithm by Spyros Reveliotis

http://www2.isye.gatech.edu/~spyros/linear_programming/linear_programming.html

Last visited: 14-02-07

OR-Notes J E Beasley

<http://people.brunel.ac.uk/~mastjjb/jeb/or/twomines>

Last visited: 16-02-07

Linear Programming

<http://home.ubalt.edu/ntsbarsh/opre640a/rpcotdp#rpcotdp>

Last visited: 21-02-07

Linear Programming Formulation

www.people.brunel.ac.uk/~mastjjb/jeb/lp.html

Last visited: 15-02-07

Detailed description of the activity

The learner will:

- Read the notes that are available in the recommended textbook in order to get a general understanding of the formulation of a Linear Programming Problem.
- Discuss in groups a situation which can be interpreted as a Linear Programming Problem and its formulation. Here one person describes a maximization/minimization situation and allows someone else to formulate that situation.
- Attempt the exercise given: The learner refers to indicated pages of the recommended textbook.
- Open the relevant links to get deeper understanding of processes of problem formulation.



Learning activities



Problem: *A manager is worried about what to do to maximise production levels.*

Suppose that we have a commercial farmer who produces a variety of farm produce on a quarterly basis which include maize, beef, beans, etc. This farmer can only produce a certain number of products due to several reasons. The main limitations are the size of his/her farm, the amount of inputs he/she needs, the market, climatic conditions, etc. Despite all these setbacks the farmer hopes to get some profit after selling his/her produce. This is a typical example of a linear programming problem situation. Before defining what we mean, in general, by a linear programming problem, let us consider a few practical real-world problems that serve to motivate and, at least vaguely, to define this subject.

Reading

Read the following textbook section:

Linear Programming: Foundations and Extensions by Robert J. Vanderbei
pages 3-10 (Managing a production facility)

Read also the following link:

The Standard Form of the Linear Programming Problem

http://en.wikipedia.org/wiki/Linear_programming

Last visited: 11-02-07

Notes

The following may help you to understand the notations from the above readings:

n is the number or variety of products produced by the company.

m is the number of types of raw materials .

b_i is the amount of i th raw material , for $i= 1, 2, \dots, m$

p_i is the unit market price of i th raw material, for $i=1, 2, \dots, m$ at a given moment.



a_{ij} is the amount of i th raw material required to produce one unit of j th product, for $j=1,2,\dots,n$

σ_j is the unit market price for the j th product, for $j=1,2,\dots,n$

x_j is the number of j th product produced

$\sum_{i=1}^m p_i a_{ij}$ is the cost of production of one of j th product,

$$\sum_{i=1}^m p_i a_{ij} = p_1 a_{1j} + p_2 a_{2j} + \dots + p_m a_{mj}$$

c_j is net profit for the j th product

$$c_j = \sigma_j - \sum_{i=1}^m p_i a_{ij}$$

$c_j x_j$ is the total net profit for x_j units of product j .

Discussion

In pairs narrate a situation in which Linear Programming is involved and then formulate that problem into a Linear Programming Problem.

Are you able to visualize the Formulation?

The General Form of the Linear Programming Problem

The general form for a Linear Programming problem is as follows:

Objective Function:

Maximise/minimise

$$f(x_1, x_2, \dots, x_n) = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$



Where:

x is the decision variable

n is the number of decision variables in the objective function

c_j is unit cost or unit profit of the j^{th} decision variable

Such that:

Technological Constraints:

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b$$

for $i = 1, 2, \dots, m$

Where

$$a_{i1} = \begin{pmatrix} a_{11} \\ M \\ a_{m1} \end{pmatrix}, a_{i2} = \begin{pmatrix} a_{12} \\ M \\ a_{m2} \end{pmatrix}, \dots, a_{in} = \begin{pmatrix} a_{1n} \\ M \\ a_{mn} \end{pmatrix}$$

Sign Restrictions

$$x_j \geq 0$$

Further reading

Read the following link:

An Introduction to Linear Programming and the Simplex Algorithm by Spyros Reveliotis

http://www2.isye.gatech.edu/~spyros/linear_programming/linear_programming.html



A Step-by-Step formulation of the linear programming problem

Question: What is linear programming problem formulation?

Answer: Problem formulation or modeling is the process of translating a verbal statement of a problem into a mathematical statement.

Example of linear programming problem [adapted from Koshy (1979)]

A farmer grows tomatoes and peas on her 125 hectares piece of land. It costs \$20 000 dollars grow a hectare of tomatoes and \$10 000 to grow a hectare of peas. However, Agri Bank gave her only \$1 500 000 as loan. It takes 18 working-hours to grow a hectare of tomatoes and 6 working-hours to grow a hectare of peas. She wants to devote 1080 working-hours for the whole job. If the profits from a hectare of tomatoes and a hectare of peas are \$40 000 and \$25 000, respectively. How many hectares of each should she grow to maximize her total profit? What is the maximum possible profit?

Steps to follow

Question: What is the problem of the farmer?

Answer: The problem of the farmer is that of maximising her profit having constraints on available land, production costs, and the number of working hours

Question: What are the constraints?

Answer:

- i) Let x_1 and x_2 be the decision variables of the number of hectares for beans and peas respectively.
Then $x_1 + x_2 \leq 125$ constraint on available land.
- ii) Her total growing expenses are $20000x_1 + 10000x_2 \leq 1500\ 000$
Or $2x_1 + x_2 \leq 150$ constraint on expenses
- iii) The total number of required working hours is $18x_1 + 6x_2 \leq 1080$
Or $3x_1 + x_2 \leq 180$ constraint on working hours
- iv) $x_1 \geq 0$, and $x_2 \geq 0$ non negativity constraints

Thus the problem is to maximise the profit function

$$f(x_1, x_2) = 40000x_1 + 25x_2 \quad (1)$$

subject to the constraints



$$\left. \begin{array}{l} x_1 + x_2 \leq 125 \\ 2x_1 + x_2 \leq 150 \\ 3x_1 + x_2 \leq 180 \\ \text{and} \\ x_1 \geq 0, x_2 \geq 0 \end{array} \right\}$$

(2)

Summary guidelines for linear programming problem formulation

- a) Understand the problem thoroughly.
- b) Describe the objective.
- c) Describe each constraint.
- d) Define the decision variables.
- e) Write the objective in terms of the decision variables.
- f) Write the constraints in terms of the decision variables.

Exercise 1

1. Formulate a mathematical model of the problems the production manager and comptroller are suppose to solve.
2. Give examples of four other fields where optimization plays a pivotal role.
3. State the three conditions that are necessary for linear programming problem to be linear.
4. Go the page 8 (of textbook) of Linear Programming: Foundations and Extensions by Robert J. Vanderbei and do questions 1.1 and 1.2



The General Linear Programming Problem

Readings

Read page 6 in the textbook.

Foundations and Extensions by Robert J. Vanderbei.

Follow this internet link for a detailed definition:

http://www.it.uu.se/edu/course/homepage/opt1/ht06/Lectures/fundamentat_thm-2up.pdf

Discussion

Find colleagues to talk this through with.

Do you now understand the formulation of a linear programming problem? If not consult your tutor or your colleagues who have grasped the main idea.

Note

The following is a summary for the formulation of a Linear Programming Problem points to take note of:

- Read the whole problem situation.
- Identify and define your unknown quantities (variables and constants).
- Express the objective function and the constraints in terms of variables.

Answers to Exercise 1

Question 1

For the answer to this question check “Managing a Production Facility” from page 3 to 5 of Linear Programming: Foundations and Extensions

Question 2

Some of the major application areas to which linear programming can be applied are:

Blending
Production planning
Oil refinery management
Distribution
Financial and economic planning
Manpower planning
Blast furnace burdening
Farm planning

**Question 3**

Conditions for a mathematical model to be a linear program (linear programming) are:

- all variables continuous (i.e. can take fractional values)
- a single objective (minimise or maximise)
- the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown.

Question 4

1.1

Let x and y be our variables for time given to produce bands and coils respectively

Time constraint

$$x + y \leq 40$$

Tonnage constraints

$$200x \leq 6000$$

$$140y \leq 4000$$

Objective function

$$f(x, y) = 25(200x) + 30(140y)$$

We rewrite the problem as:

Maximize

$$f(x, y) = 5000x + 4200y$$

Such that

$$x + y \leq 40$$

$$200x \leq 6000$$

$$140y \leq 4000$$

$$x \geq 0, y \geq 0$$



Learning Activity 2 : A Geometrical Approach to Linear Programming

Specific Learning Objectives

By the end of this section the learner must:

- Be able to identify and construct linear equations.
- Be able to construct the wanted regions given a set of inequalities, i.e. the feasible regions of each Linear Programming Problem.
- Be able to solve Linear Programming Problems using the geometric approach.

Summary of the Learning Activity

In this activity we introduce the resolution of linear programming problems by geometry. Here we look at notions like feasible region, feasible solutions, optimal solution and convexity. Geometrically, linear constraints define a convex polyhedron, which is called the feasible region. Since the objective function is also linear, hence a convex function. The linearity of the objective function also implies that an optimal solution can only occur at a boundary point of the feasible region, unless the objective function is constant, when any point is a global minimum.

Note: There are two situations in which no optimal solution can be found. First, if the constraints contradict each other (for instance, $x \geq 2$ and $x \leq 1$) then the feasible region is empty and there can be no optimal solution, since there are no solutions at all. In this case, the linear programming is said to be infeasible.

Key Terms (refer to glossary)

Feasible region

Feasible solutions

Optimal solution and convexity

Feasible production set

Production possibility set or opportunity set

Extreme points

Infeasible problem

Hyperplane

Halfspace

Convex polyhedral set

Convex polyhedral cone



List of Required Readings

Linear Programming: Foundations and Extensions by Robert J. Vanderbei

www.princeton.edu/~rvdb/LPbook/onlinebook.pdf

Last visited: 15-02-07

Lecture notes on optimization by Pravin Varaiya

http://robotics.eecs.berkeley.edu/~varaiya/papers_ps.dir/NOO.pdf

Last visited: 10-02-07

List of Relevant Useful Links

Linear Programming: A Geometric Approach

<http://www.wiley.com/college/sc/sullivan/CH03.pdf>

Last visited: 16-02-07

Graphical Solution Method

<http://home.ubalt.edu/ntsbarsh/opre640a/rpcotdp#rpcotdp>

Last visited: 21/02/07

Linear Programming: Geometric Approach

www.math.tamu.edu/~janice.epstein/141/notes/Ch3.pdf

Last visited: 16-02-07

Detailed Description of the Activity

The learner will:

- Read from the link given in order to have an understanding of the resolution of a Linear Programming problem by geometry. This will give the learner a general picture on the geometry approach.
- Analyze the example given in the link. This will help the learner to understand the steps and procedures followed in the resolution.
- Do the exercise from the textbook Linear Programming: Foundations and Extensions by Robert J. Vanderbei.



Learning Activities



A man walking along a line-segmented boundary of a field

Reading

Linear Programming: Foundations and Extensions by Robert J. Vanderbei page 22 section 5.

www.princeton.edu/~rvdb/LPbook/onlinebook.pdf

Last visited: 15-02-07

Further reading

Read the following link on the Geometrical Approach

Linear Programming: A Geometric Approach page 171-176

<http://www.wiley.com/college/sc/sullivan/CH03.pdf>

Last visited: 16-02-07

This link will help you to understand how to graph inequalities and identify the wanted regions which are called feasible regions in linear programming problems. The feasible region contains all possible solutions and from these we find the optimal solution.

NOTE

When the number of variables in a linear programming problem is two or three the feasible solutions can be determined graphically by drawing the graphs of inequalities of the constraints.

The following steps are essential when one is solving linear programming problems geometrically.

Step 1: Graph the constraints

Step 2: Identify the feasible region

Step 3: Graph the objective function (twice)

Step 4: Find the cornerpoints to the region of feasible solutions

Step 5: Evaluate the objective function at all the feasible corner points by:



Step 6: Identifying the optimal point

Step 7: Finding the coordinates of the optimal point

Step 8: Evaluating the objective function at the optimal solution.

Example

Lets us try to follow the steps using the following problem.

$$f(x, y) = 5x + 7y$$

Maximize:

subject to:

$$x + y - 7 \leq 0$$

$$2x - 3y + 6 \geq 0$$

$$x \geq 0, y \geq 0$$

1. Is the problem amenable to Linear Programming?

Yes, if and only if: All variables have power of 1, and they are added or subtracted (not divided or multiplied). The constraint must be of the following forms (\leq , \geq , or $=$, that is, the linear programming-constraints are always closed), and the objective must be either maximization or minimization.

Answer Yes

2. Can I use the graphical method?

Yes, if the number of decision variables is either 1 or 2.

3. Use Graph Paper. Graph the constraints:

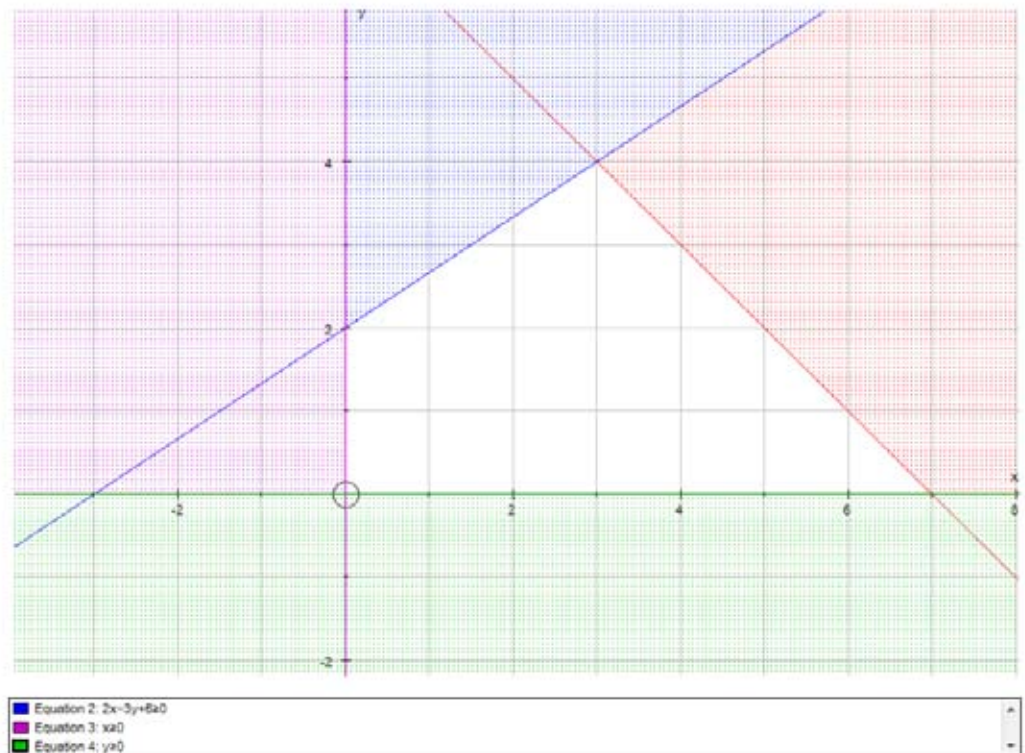
$$x + y - 7 \leq 0$$

$$2x - 3y + 6 \geq 0$$

$$x \geq 0, y \geq 0$$



4. Shade the unwanted region of each constraint inequality.



5. Throw away the sides that are not feasible.

After all the constraints are graphed, you should have a non-empty (convex) feasible region, unless the problem is infeasible.

6. Create (at least) two iso-value lines from the objective function, by setting the objective function to any two distinct numbers. Graph the resulting lines. By moving these lines parallel, you will find the optimal corner (extreme point), if it does exist.

In general, if the feasible region is within the first quadrant of the coordinate system then, for the maximization problems you are moving the iso-value objective function parallel to itself *far away from the origin point* $(0, 0)$, while having at least a common point with the feasible region. However, for minimization problems the opposite is true, that is, you are moving the iso-value objective function parallel to itself *closer to the origin point*, while having at least a common point with the feasible region. The common point provides the optimal solution.

**Exercise 2**

Read Linear Programming: Foundations and Extension by Robert J. Vanderbei page 24 Work through numbers 2.1, 2.2, 2.3, 2.5 and 2.10.

Note: You are only required to construct the feasible regions

Further reading

Now go to the following links and read more on resolution by geometry:

Linear Programming: A Geometric Approach page 171-176

<http://www.wiley.com/college/sc/sullivan/CH03.pdf>

This is a maximizing problem in which the profit is being maximized.

Linear Programming: Geometric Approach

www.math.tamu.edu/~janice.epstein/141/notes/Ch3.pdf

Last visited: 16-02-07

Answers to Exercise 2

For answers to the given problems check on page 449 of your main book, Linear Programming: Foundations and Extensions by Robert J. Vanderbei



UNIT 2: RESOLUTION AND ANALYSIS

Learning Activity # 3: Optimality Conditions

Specific Learning Objectives

After studying this section, you should:

- be able to interpret the derivatives and existence of optimality
- be able to prove the theorems of optimality
- be able to explain and prove the The Fundamental Theorem of Linear Programming
- be able to check optimality conditions for solving a Linear Programming problem.
- be able to carry out a sensitivity analysis for a given optimal solution.

Summary of the Learning Activity

In this unit we want to consider the computational aspects of the linear programming problem and do a sensitivity analysis used to determine how stable the optimal solution is to changes in some related variables. Here the resolution is done algebraically, i.e. using the notions of duality, simplex method, the method of Big M, and the mutual primal algorithm. Therefore the learner needs knowledge from unit 1 in order to understand unit 2 because the transformation of the linear programming problem into standard form helps in the resolution of the linear programming problem algebraically. In this unit we do some analysis using the derivatives and existence conditions of optimality.

Key Terms (refer to glossary)

Basic variables

Nonbasic variables

Augmented matrix

Pivot

Pivot column

Test ratio

Tableau

Cycling



List of Required Readings

Linear Programming: Foundations and Extensions by Robert J. Vanderbei

www.princeton.edu/~rvdb/LPbook/onlinebook.pdf

Last visited: 15-02-07

Lecture notes on optimization by Pravin Varaiya

http://robotics.eecs.berkeley.edu/~varaiya/papers_ps.dir/NOO.pdf

Last visited: 10-02-07

List of Relevant Useful Links

Optimality Conditions for Constrained Optimization

http://ocw.mit.edu/NR/rdonlyres/Sloan-school-of-Management/15-084JSpring2004/7240EF84-B20D-419F-B1C0-2DAF3277F5C4/0/lec6_constr_opt.pdf

Last visited: 16-02-07

Optimality Conditions

<http://www.math.mtu.edu/%7Emsgocken/ma5630spring2003/lectures/lag1/lag1/node1.html>

Last visited: 16-02-07

Alternate Optimal Solutions, Degeneracy, Unboundedness, Infeasibility

<http://mat.gsia.cmu.edu/QUANT/notes/node63.html#SECTION00830000000000000000>



Learning Activities



“How do you I know I have arrived at the best position?”

Optimality Conditions

In simple terms, optimality conditions are certain conditions which should tell you that you have reached the optimal solution. For the Maximization Problem if all nonbasic variables have negative or zero coefficients in the objective function, an optimal solution has been obtained. If you were to substitute one of the non- basic variables with a non-negative value (since the variables are non-negative), then the value of the final solution is reduced. Hence you are certain that the optimal solution has been obtained. Similarly, for the Minimization Problem if all non-basic variables have positive or zero coefficients in the objective function, an optimal solution has been obtained.

Reading

For Conditions of Optimality for a Linear Programming Problem and the Fundamental Theorem of Linear Programming, read

Linear Programming: Foundations and Extensions by Robert J. Vanderbei

www.princeton.edu/~rvdb/LPbook/onlinebook.pdf

Last visited: 15-02-07

Further Reading

<http://enr.smu.edu/~barr/ip/ch1/node7.html>

<http://www.maths.abdn.ac.uk/~igc/tch/mx3503/notes/node67.htm>



Exercise 3

Look at the example from Linear Programming: Foundations and Extension by Robert J. Vanderbei, page 13, whose solution is given on pages 15 and 16.

Now do this Exercise adapted from Wagner (1975)

Test for optimality conditions on the following linear programming problem

$$\text{Maximize } 4x_1 + 5x_2 + 9x_3 + 11x_4$$

Subject to

$$x_1 + x_2 + x_3 + x_4 \leq 15$$

$$7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 120$$

$$3x_1 + 5x_2 + 10x_3 + 15x_4 \leq 100$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0$$

Solution

Let x_0 be the value of the objective function add the slack variables x_5, x_6, x_7 to your constraints so that you have equality. Then write the system as

$$x_0 - 4x_1 - 5x_2 - 9x_3 - 11x_4 = 0$$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 15$$

$$7x_1 + 5x_2 + 3x_3 + 2x_4 + x_6 = 120$$

$$3x_1 + 5x_2 + 10x_3 + 15x_4 + x_7 = 100$$

After going through four iterations of the Simplex Method you get to a stage where the nonbasic variables x_2, x_4, x_5 or x_7 can only have value zero. That is to say we have

$$x_0 = \frac{695}{7} - \frac{3}{7}x_2 - \frac{11}{7}x_4 - \frac{13}{7}x_5 - \frac{5}{7}x_7$$

This is the optimality condition which allows us to terminate the iteration.



Learning Activity 4: Algebraic Approach

Specific Learning Objectives

After studying this section, you will:

- Be able to interpret the derivatives and existence of optimality.
- Be able to apply the Method (penalty) of big M and Dual simplex method.
- Be able to differentiate the properties of duality, mutual primal algorithm, and dual simplex algorithm.
- Be able to follow the given algorithms step-by-step, i.e. mutual primal and the dual simplex algorithms.
- Be able to solve linear programming problems using appropriate mathematical software (when available)

Summary of the Learning Activity

In this section we will learn how to solve the linear programming problem algebraically. The main algorithm we shall follow is the Simplex Algorithm. The simplex algorithm, developed by George Dantzig (1947) solves linear programming problems by constructing an admissible solution at a vertex of the polyhedron and then walking along edges of the polyhedron to vertices with successively higher values of the objective function until the optimum is reached. Although this algorithm is quite efficient in practice and can be guaranteed to find the global optimum if certain precautions against cycling are taken, it has poor worst-case behavior: it is possible to construct a linear programming problem for which the simplex method takes a number of steps exponential in the problem size. Actually, for some time it was not known whether the linear programming problem was solvable in polynomial time.

Key Terms (refer to glossary)

Basis

Basic variables

Nonbasic variables

Slack variable

Surplus variable

Artificial variable

Basic solution

Degeneracy



Primal problem
Dual problem
Weak duality
Strong duality
Primal feasibility
Dual feasibility
Primal feasibility
Dual feasibility

List of Required Readings

Linear Programming: Foundations and Extensions by Robert J. Vanderbei

www.princeton.edu/~rvdb/LPbook/onlinebook.pdf

Last visited: 15-02-07

Lecture notes on optimization by Pravin Varaiya

http://robotics.eecs.berkeley.edu/~varaiya/papers_ps.dir/NOO.pdf

Last visited: 10-02-07

List of Relevant Useful Links

Simplex Method - Big M

http://www.math.uwo.ca/~heinicke/courses/236_03/bigM.pdf

Last visited: 15-02-07

Simplex Method – Big M

<http://www.computing.dcu.ie/~lkillen/teach/CA427Simplexbigmexample.pdf>

Last visited: 15-02-07

Dual Problem: Construction and Its Meaning

<http://home.ubalt.edu/ntsbarsh/opre640a/rpcotdp#rpcotdp>

Sensitivity Analysis for Linear Programming

<http://mat.gsia.cmu.edu/QUANT/notes/node64.html>

Last visited: 20/02/07



Detailed Description of the Activity

The learner will:

- Read the the textbook Linear Programming: Foundations and Extensions on the Simplex Method pages 13-28, 29-43, and 89-107 of the textbook
- Analyze the example that is provided in the link below:
 - Simplex Method - Big M
http://www.math.uwo.ca/~heinicke/courses/236_03/bigM.pdf
Last visited: 15-02-07
- Do the exercise from the textbook Linear Programming: Foundations and Extensions on the Simplex Method page 13 of the textbook numbers 2.1, 2.2, 2.3 and 2.4.

In addition to reading about the logic of the Simplex Algorithm and practicing applying it to specific linear programming problem situations, in this Learning Activity the learner will be encouraged to also perform those Simplex computational procedures using appropriate software that is accessible to the learner.



Learning Activities



A woman, baby at the back, pondering over the easiest way of crossing a fast flowing river.

The Simplex Method

The methods of linear programming are based on the theories of matrices and finite-dimensional vector spaces and the simplex method for solving linear programming problems is built around the basic solution of a set of simultaneous linear equations. For more details about the basic solutions and vector spaces read your module on Linear algebra or try the well illustrated example by Richard S. Barr on the following link: <http://engr.smu.edu/~barr/ip/ch1/node6.html>

When you are solving linear programming problems using the Simplex algorithm the main steps to remember are:

1. Set up the initial simplex tableau
2. Locate the pivot of the **tableau**
3. If the pivot is 1, then go to step 4 ; otherwise divide the pivotal row by the pivot to get 1 in the pivotal position
4. Convert the remaining entries of the pivotal column into zeros by using the elementary row operations (which you have learnt in the linear algebra module)
5. Repeat steps 2 to 4 until a tableau with non-negative indicators is obtained. This is the final tableau we need in this algorithm and it is called the **terminal simplex**.
6. The optimal solution and the maximum value of the objective function can be read from that terminal tableau.

Let us go through the steps using the following example adapted from by Koshy (1979)

Maximize

$$f(x, y) = 170x + 225y$$



Subject to:

$$x + y \leq 300$$

$$2x + 3y \leq 720$$

$$x \geq 0, y \geq 0$$

Step 1 Form the initial simplex tableau

Let us introduce two slack variables $s_1 \geq 0$ and $s_2 \geq 0$ then the first inequalities can be written as equalities

$$x + y + s_1 = 300$$

$$2x + 3y + s_2 = 720$$

We can now have the system of equations

$$x + y + s_1 = 300$$

$$2x + 3y + s_2 = 720$$

$$-170x - 225y + f = 0$$

Forming the initial simplex tableau

x	y	s₁	s₂	f	rhs
1	1	1	0	0	300
2	3	0	1	0	720
-170	-225	0	0	1	0

Step 2 Locate the pivot of the tableau

x	y	s₁	s₂	f	rhs
2	1	1	0	0	300
2	3	0	1	0	720
-170	-225	0	0	1	0

Pivot row

Pivot

Pivot column



Step 3 Since the pivot is 3, we divide the second row by 3

x	y	s₁	s₂	f	rhs
1	1	1	0	0	300
2/3	1	0	1/3	0	240
-170	-225	0	0	1	0

Step 4 Adding -1 times the second to the first and 225 times the second row to the third, we get

x	y	s₁	s₂	f	rhs
1/3	0	0	-1/3	0	60
2/3	1	0	1/3	0	240
-20	0	0	75	1	5400

New pivot

Step 5 The new pivot is 1/3; we divide the first row by 1/3 and then add -2/3 times the resulting first row to the second and 20 times the first row to the third row.

x	y	s₁	s₂	f	rhs
1	0	3	-1	0	180
0	1	-2	1	0	120
0	0	60	55	1	57600

Maximum value

Step 6 This tableau is the augmented matrix of the system

$$x + 3s_1 - s_2 = 180$$

$$y - 2s_1 + s_2 = 120$$

$$60s_1 + 95s_2 = 57\,600$$

That is

$$x = 180 - 3s_1 + s_2$$

$$y = 120 + 2s_1 - s_2$$

$$f = 57600 - 60s_1 - 95s_2$$



f has maximum value of 57 600 when $s_1 = 0$ and $s_2 = 0$, and this yields $x = 180$ and $y = 120$

Note

The values of f , x and y can be easily read from the terminal simplex, thus the optimal solution and the maximum value of the objective function can be read from this terminal tableau.

The Big M method

In the case where we cannot find an initial basic feasible solution you could use the Big M method where you add a non-negative large integer $M > 0$.

To illustrate the use of the Big M method, let us follow the example adapted from Wagner (1975)

$$\text{Maximize } f(x_1, x_2) = -3x_1 - 2x_2$$

Subject to

$$x_1 + x_2 = 10$$

$$x_1 \geq 4$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

Then after adding a surplus variable x_3 in the inequality above, you can write the model as

$$x_0 + 3x_1 + 2x_2 = 0$$

$$x_1 + x_2 = 10$$

$$x_1 - x_3 = 4$$

Where $f(x_1, x_2) = -3x_1 - 2x_2$ has been written as

$$x_0 + 3x_1 + 2x_2 = 0 \quad \text{where } x_0 = f(x_1, x_2)$$



Next, introduce artificial variables y_1 and y_2 , and let $M=10$ as our large integer, for instance, giving

$$x_0 + 3x_1 + 2x_2 + 10y_1 + 10y_2 = 0$$

$$x_1 + x_2 + y_1 = 10$$

$$x_1 - x_3 + y_2 = 4$$

To initiate the algorithm, you have to subtract ($M=10$) times row 2 and ($M=10$) times row 3 from row 1 to eliminate y_1 and y_2 :

$$x_0 - 17x_1 - 8x_2 + 10x_3 = -140$$

$$x_1 + x_2 + y_1 = 10$$

$$x_1 - x_3 + y_2 = 4$$

From here you can then proceed using the Simplex method.

You can verify that $x_1 = 4$ and $x_2 = 6$ are optimal.

Further reading

Follow this link.

The Simplex Method:

<http://www-fp.mcs.anl.gov/otc/Guide/CaseStudies/simplex/feasible.html>

Further Activity

The following link <http://www-fp.mcs.anl.gov/otc/Guide/CaseStudies/simplex/feasible.html> explains the simplex method in a user friendly way, go through the activities in it and then answer the following questions:

1. State the nature of a standard maximising problem
2. What happens in the initial simplex tableau stage?



Exercise 4

The following tableaux tables were obtained in the course of solving linear program non-negative variables x_1 and x_2 and 2 inequality constraints, the objective function, z is maximisation. Slack variables s_1 and s_2 were added. In each case, indicate whether the linear program:

- (i) is unbounded
- (ii) has a unique optimum solution
- (iii) has an alternate optimum solution
- (iv) is degenerate (in this case, indicate whether any of the above holds).

a)

z	x_1	x_2	s_1	s_2	rhs
1	0	3	2	0	20
0	1	-2	-1	0	4
0	0	-1	0	1	2

b)

z	x_1	x_2	s_1	s_2	rhs
1	0	-1	0	2	20
0	0	0	1	-2	5
0	0	-2	0	3	6

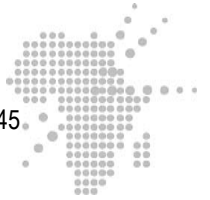
c)

z	x_1	x_2	s_1	s_2	rhs
1	2	0	0	1	8
0	3	1	0	-2	4
0	-2	0	1	1	0

d)

z	x_1	x_2	s_1	s_2	rhs
1	0	0	2	0	5
0	0	-1	1	1	4
0	1	-1	-1	0	4

Hint: Answers to Exercise 4 can easily be obtained by referring to the definitions of “unboundedness,” “unique optimum solution,” “alternate optimum solution,” and “degeneracy.”



Formulation of the Dual Problem

Every linear programming has another linear programming called its **dual**, which shares the same data and is derived through rational arguments. In this context the original linear programming is called the **primal** linear programming. Variables in the dual problem are different from those in the primal, each dual variable is associated with a primal constraint, it is the marginal value or Langrange multiplier corresponding to that constraint.

The problem faced by the Production Manager as the Optimist and the Comptroller as the Persimist in Linear Programming: Foundations and Extensions by Robert J. Vanderbei, chapter 1, identifies a situation for a dual problem. The Resource Allocation Problem in chapter 5 pages 73-78 of same book is a good illustration of the formulation of a dual problem from its primal problem.

Example [adapted from Arsham (2007) [<http://home.ubalt.edu/ntsbarsh/opre640a/partVIII.htm>)]

Let us illustrate the formulation of the dual problem using a tailor's problem

Uncontrollable Inputs

	trousers	shirts	available
labour	3	2	50
raw material	4	3	60
net income	20	15	

and its linear programming is:

Maximize

$$f(x_1, x_2) = 20x_1 + 15x_2$$

subject to:

$$3x_1 + 2x_2 \leq 50 \quad \text{labour constraint}$$

$$4x_1 + 3x_2 \leq 60 \quad \text{material constraint}$$

$$x_1 \geq 0, x_2 \geq 0$$

Where x_1 and x_2 are the number of trousers and shirts to make.

Suppose the tailor wishes to buy insurance for his net income. Let z_1 = the dollar amount payable to the tailor for every labour hour lost due to unforeseen problems, and z_2 = the dollar amount payable to the tailor for every raw material unit lost.



The insurance broker tries to minimize the total amount of $\$(50z_1 + 60z_2)$ payable to the tailor by the insurance company. However, the tailor will set the constraints (conditions) by insisting that the insurance company cover all his loss (that is, his net income) since he cannot make the products. Therefore the broker's problem is:

Minimize

$$f(z_1, z_2) = 50z_1 + 60z_2$$

subject to:

$$3z_1 + 4z_2 \geq 20 \quad \text{Net income from a trouser}$$

$$2z_1 + 3z_2 \geq 15 \quad \text{Net income from a shirt}$$

$$z_1 \geq 0, z_2 \geq 0$$

If we work out the solutions of this problem you will see that the insurance company's problem is closely related to the tailor's problem.

Reading

Now read the following text:

Linear Programming: Foundations and Extensions by Robert J. Vanderbei, Chapter 5, pages 55 – 79. [www.princeton.edu/~rvdb/LPbook/onlinebook.pdf]

Last visited: 15-02-07

Dual Theorem

- a) In the event that both primal and dual problems possess feasible solutions, then the primal problem has an optimal solution x_j^* for $j = 1, 2, \dots, n$, the dual problem has an optimal solution y_i^* , for $i = 1, 2, \dots, m$, and

$$\sum_{j=1}^n c_j x_j^* = \sum_{i=1}^m b_i y_i^*$$

- b) If either the primal or dual problem possesses a feasible solution with a finite optimal objective-function value, then the other problem possesses a feasible solution with the optimal objective-function value.



Note

The duality relationship can be summarised as follows:

Primal (Maximize)	Dual (Minimize)
Objective function	Right-hand side
Right-hand side	Objective function
j^{th} column of coefficient	j^{th} row of coefficient
i^{th} row of coefficients	i^{th} column of coefficient
j^{th} variable nonnegative	j^{th} relation an inequality (\geq)
j^{th} variable unrestricted in sign	j^{th} relation an equality
i^{th} relation an inequality (\leq)	i^{th} variable nonnegative
i^{th} relation an equality	i^{th} variable unrestricted in sign

Example [adapted from Wagner (1975)]

Here is a didactic example to illustrate the formation of a dual problem from its primal.

Maximize

$$4x_1 + 5x_2 + 9x_3 + 11x_4$$

Subject to:

$$x_1 + x_2 + x_3 + x_4 \leq 15$$

$$7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 120$$

$$3x_1 + 5x_2 + 10x_3 + 15x_4 \leq 100$$

$$x_j \geq 0$$

For $j=1, 2, 3, 4$

The dual problem can easily be obtained using the duality relationships in the table above as:

Minimize

$$15y_1 + 120y_2 + 100y_3$$

Subject to

$$x_1 + x_2 + x_3 + x_4 \leq 15$$

$$7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 120$$

$$3x_1 + 5x_2 + 10x_3 + 15x_4 \leq 100$$

$$x_j \geq 0$$



Exercise 5

1. In your own words explain what you understand by a dual problem in linear programming.
2. State the dual theorem and then prove it.
3. Do questions 5.1, 5.2 and 5.16 on page 79 of Linear Programming : Foundations and Extensions by Vanderbei.

Further Reading

You can read the following link for more examples of formulating a dual problem in linear programming: <http://www.maths.abdn.ac.uk/~igc/tch/mx3503/notes/node66.html>

Duality in linear programming, Optimality Conditions

<http://www-personal.umich.edu/~murty/310/310-2slides8.pdf> -

Duality and Optimality Conditions

<http://www.caam.rice.edu/~yzhang/caam378/Notes/chap5.pdf>

Answers to Exercise 5

Question 1

Read the Tailor's Problem or the Resource Allocation Problem and then explain them in your words to your colleague.

Question 2

Weak and strong duality theorems combined together.

Question 3

For question 5.1 we are given the problem as follows

Maximize $x_1 - 2x_2$

Subject to

$$x_1 + 2x_2 - x_3 + x_4 \geq 0$$

$$4x_1 + 3x_2 + 4x_3 - 2x_4 \leq 3$$

$$-x_1 - x_2 + 2x_3 + x_4 = 1$$

$$x_2 \geq 0, x_3 \geq 0$$

We rewrite the problem as follows



Maximize $x_1 - 2x_2$

Subject to

$$-x_1 - 2x_2 + x_3 - x_4 \leq 0$$

$$4x_1 + 3x_2 + 4x_3 - 2x_4 \leq 3$$

$$-x_1 - x_2 + 2x_3 + x_4 \leq 1$$

$$x_1 + x_2 - 2x_3 - x_4 \leq -1$$

The dual of this problem is therefore:

Minimize $3y_2 + y_3 - y_4$

Subject to

$$-y_1 + 4y_2 - y_3 + y_4 \geq 1$$

$$2y_1 + 3y_2 - y_3 + y_4 \geq -2$$

$$y_1 + 4y_2 + 2y_3 - 2y_4 \geq 0$$

$$-y_1 - 2y_2 + y_3 - y_4 \geq 0$$

5.2 To solve this problem refer to theorem 5.2 on page 60 of *Linear Programming: Foundations and Extensions*. Find the dual of the problem as illustrated in the problem above.



Learning activity 5: Sensitivity Analysis

Specific Learning Objective

By the end of the activity the learner:

- should be able to carry out a sensitivity analysis of a Linear Programming solution.
- Should be able to explain the reasons for carrying a sensitivity analysis.

Summary of the Learning Activity

In this activity we are going to answer some of the post-optimality questions such as

- 1) If the profit contribution of a particular basic activity decreases, does the current solution remain optimal?
- 2) What happens if resource availability is curtailed?
- 3) What happens if a new activity is added?
- 4) How far the input parameter values vary without causing violent changes in an optimal solution.

We are also going to do marginal and parametric analysis and learn what is involved in each of the analyses. Learners should to do some sensitivity analysis on some optimal solutions they have found in activity 4.

Key Terms (refer to glossary)

Parametric analysis

Marginal analysis

List of Required Readings:

Linear Programming: Foundations and Extensions by Robert J. Vanderbei

www.princeton.edu/~rvdb/LPbook/onlinebook.pdf

Last visited: 15-02-07

Lecture notes on optimization by Pravin Varaiya

http://robotics.eecs.berkeley.edu/~varaiya/papers_ps.dir/NOO.pdf

Last visited: 10-02-07



List of Relevant Useful Links

Sensitivity Analysis

<http://www.jr2.ox.ac.uk/bandolier/booth/glossary/sensanal.html>

Last visited: 15-02-07

Sensitivity Analysis

http://pespmc1.vub.ac.be/ASC/SENSIT_ANALY.html

Last visited: 15-02-07

Sensitivity Analysis

http://en.wikipedia.org/wiki/sensitivity_analysis

Last visited: 15-02-07

Management Science: Linear Programming Notes

<http://www.strathcona.bham.ac.uk/Pdfs1%20management%20Course%20Year%202/LINEAR%20PROGRAMM%20NOTES.PDF>

Detailed Description of the Activity

The learner will read on the definition of sensitivity analysis and why it is carried out. They will refer mainly to the recommended text: Linear Programming: Foundations and Extensions by Robert J. Vanderbei, which explains the purpose of sensitivity analysis well. The learner is encouraged to discuss with colleagues and perform some practical computations associated with sensitivity analysis using appropriate (and accessible) software packages. Interpreting the results of running a sensitivity analysis is also an important exercise as it has a direct bearing on eventual decision making processes. A number of links that discuss this issue of sensitivity analyses and provide examples of software based computer runs are supplied for the learner in this Learning Activity. The learner will increase her/his appreciation and understanding if he/she also checks on these links.

Learning Activities

Consider the problem of the production manager that we discussed in Activity One. Even under normal business situations, various factors influence the production, or, eventually, profit levels. Suppose the manager has been advised by her planning committee that has solved the associated linear programming problem to maintain purchase orders of some input materials and supplies to some warehouses at certain specific levels. Those levels would be corresponding to the optimal solution (production or profits) under those circumstances. Now consider the case of an inflationary environment in a country with a poorly performing economy. The influencing factors will obviously vary in their impact to the production process depending on the availability and costs of the input materials and output distribution costs. The variations



are often non-uniform across factors and unpredictable in both time period and size. Under those circumstances, the manager would be very interested in knowing how much changes in individual or combinations of input factors impact on the current optimum solution. This is where sensitivity analysis, which looks at that problem, becomes useful. Technically, it analyses changes in the coefficients of the objective function (the unit costs) or the right hand sides of the constraints (the requirements) affect the optimal solution.



Discussion [adapted from Trick (2007)]

<http://mat.gsia.cmu.edu/QUANT/notes/node64.html>

Q. Where do I start in sensitivity analysis?

A. If we carry out the simplex method iterations on the initial tableau below

f	x	y	s₁	s₂	RHS
0	3	1	1	0	300
0	2	3	0	1	720
1	-170	-225	0	0	0

We get we get the final tableau called **terminal tableau** below

basis (identity matrix)					
z	x₁	x₂	s₁	s₂	RHS
1	0	0	3	-1	180
0	1	0	1	-2	120
0	0	1	60	55	57600

← Row 0

It has all non-negative values in Row 0 (which we will often refer to as the *cost row*), all non-negative right-hand-side values, and a basis matrix embedded.

Sensitivity analysis starts at this point to determine the effects of small changes in the optimal solution. We will try to determine how that change affects the final tableau, and try to re-form the final tableau accordingly.



Changing the values of the decision variables

The first change we will consider is that of changing a cost value, that is, changing the values of x and y by a small amount ϵ in the original problem. If we carry out the calculation to find the optimal solution for the changed value of x in the original problem, we will get the same final tableau as before, except that the corresponding cost entry would be ϵ lower (this is because we never did anything except add or subtract scalar multiples of Rows 1 through m to other rows; we never added or subtracted Row 0 to other rows).

f	x	y	s₁	s₂	RHS
0	3+ ϵ	1	1	0	300
0	2	3	0	1	720
1	-170	-225	0	0	0

Carry out the calculation and then confirm the statement above.

Right-hand-side changes

For these types of changes, we concentrate on maximization problems with all \leq constraints. Other cases are handled similarly.

Consider the following problem:

$$\text{Maximize } f(x, y) = 4x + 5y$$

subject to:

$$2x + 3y \leq 12$$

$$x + y \leq 5$$

$$x, y \geq 0$$

The optimal tableau, after adding slacks s_1 and s_2 is

z	x	y	s₁	s₂	RHS
1	0	0	1	2	22
0	0	1	1	-2	2
0	1	0	-1	3	3

Now suppose we change the amount of the right-hand-side from 12 to $(12+\epsilon)$ in the first constraint, the tableau changes to:



z	x	y	s₁	s₂	RHS
1	0	0	1	2	22 + ϵ
0	0	1	1	-2	2 + ϵ
0	1	0	-1	3	3 - ϵ

This represents an optimal tableau as long as the right hand side is all non-negative. In other words, we need ϵ between -2 and 3 in order for the basis not to change. For any ϵ in that range, the optimal objective will be $(22+\epsilon)$. For example, with ϵ equals 2, the new objective is 24 with $y = 4$ and $x = 1$.

Similarly, if we change the right-hand-side of the second constraint from 5 to $(5 + \epsilon)$ in the original formulation, we get an objective of $(22 + 2\epsilon)$ in the final tableau, as long as $-1 \leq \epsilon \leq 1$

Basically sensitivity analysis tests what happens to the optimal solution if there are changes in the supply of raw material or market values. It highlights changes in variables which affect the optimal solution and those which do not.

Reading and Discussion

Read and discuss with a colleague

Linear Programming: Foundations and Extensions by Robert J. Vanderbei Chapter 7 pages 111-120: www.princeton.edu/~rvdb/LPbook/onlinebook.pdf

Last visited: 15-02-07

Sensitivity Analysis

http://en.wikipedia.org/wiki/sensitivity_analysis

Last visited: 15-02-07

Use accessible software packages (such as Pivot Tool) to perform some sensitivity analyses on any 2 optimum solutions to a linear programming problem that you have previously obtained. Interpret the results of the analyses you have done and discuss your interpretations with a colleague.

Pivot Tool: <http://campusgi.princeton.edu/~rvdb/JAVA/network/nettool/netsimp.html>



Exercise 6

Linear Programming: Foundations and Extensions by Robert J. Vanderbei Chapter 7 page 120 # 7.1 and 7.2. www.princeton.edu/~rvdb/LPbook/onlinebook.pdf
Last visited: 15-02-07

Further Activity

Optimization Methods Lecture 7 Sensitivity Analysis

<http://ocw.mit.edu/NR/rdonlyres/Sloan-School-of-Management/15-093Fall-2004/8842223F-9C8A-4B9E-83F8-DD28DE270140/0/lecture07.pdf>

Management Science: Linear Programming Notes

<http://www.strathcona.bham.ac.uk/Pdfs1%20management%20Course%20Year%202/LINEAR%20PROGRAMM%20NOTES.PDF>

Answers to Exercise 6

Page 449 of Foundations and Extensions by Robert J. Vanderbei



XI. Compiled List of all Key Concepts (Glossary)

Argumented matrix

is formed by the coefficient of the dictionary of an linear programming problem

Artificial Variable

A variable added to a linear program in phase 1 to aid finding a feasible solution.

Basic Solution

x of $(Ax = b)$ is a *basic solution* if the n components of x can be partitioned into m «basic» and $n-m$ «non-basic» variables in such a way that:

- the m columns of A corresponding to the basic variables form a nonsingular basis and,
- the value of each "non-basic" variable is 0.

The constraint matrix A has m rows (constraints) and n columns (variables).

Basis

The set of basic variables.

Basic Variables

A variable in the basic solution (value is not 0).

Constraints

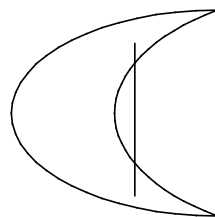
A set of equalities and inequalities that the feasible solution must satisfy.

Convex

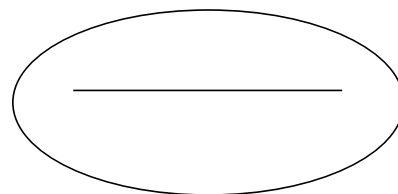
is a set of points S in the n -dim space, such that if the line segment connecting any two points $X_1, X_2 \in S$, belongs completely in S . Mathematical definition of convexity is expressed as:

$$S \text{ convex} \Leftrightarrow \forall X_1, X_2 \in S, \forall x \in (0,1) : (1-x)X_1 + xX_2 \in S$$

Convex sets



Nonconvex



Convex



Cycling

is when a sequence of pivots goes through the same tableaux and repeats itself indefinitely.

Degeneracy

occurs when some basic variable is at one of its bound values (canonically zero). Without any given qualification, a (basic) solution is degenerate if one or more of its basic values is zero (the canonical lower bound).

Dictionary

is the systems of equations of the objective function and the constraint equations after introducing slack variables

Dual degenerate

is a solution where one of its non-basic variables has zero reduced cost. In general, a solution is degenerate if it is not strictly complementary.

Duality

Weak duality is say, if x, p are feasible solutions to primal and dual problems respectively, then $b'p \leq c'x$.

Strong duality is when the primal problem has an optimal solution and the dual problem has also an optimal solution and $c'x^* = b'p^*$ [where x^* and p^* are optimal solutions of the primal and the dual problem respectively.]

Feasible

A point is feasible if it satisfies all constraints. The *feasible region* (or *feasibility region*) is the set of all feasible points. A mathematical program is feasible if its feasible region is not empty.

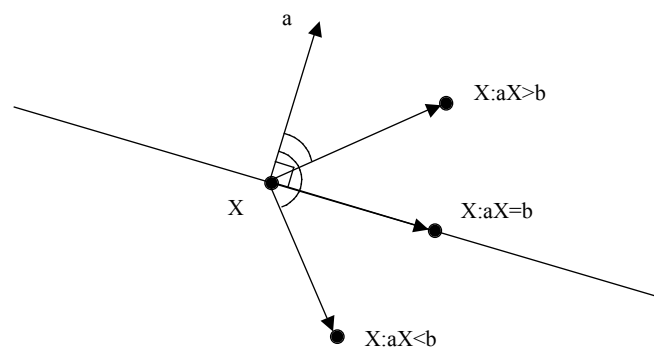
Primal feasibility

means having a feasible solution of the linear programming problem

Dual feasibility means having a feasible solution of the dual linear programming problem

Feasible Solution

A solution vector, x , which satisfies the constraints.





Half-planes

the feasible region of a linear inequality

For example the half-planes of the inequality $a_1 X_1 + a_2 X_2 \begin{pmatrix} \leq \\ \geq \end{pmatrix} b$ are the planes on

the left and right side of the equality $a_1 X_1 + a_2 X_2 \begin{pmatrix} \leq \\ \geq \end{pmatrix} b$.

given a linear constraint:

$$a_1 X_1 + a_2 X_2 + \dots + a_n X_n \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b \dots \dots \dots **$$

$$X_0 = \langle X_{01}, X_{02}, \dots, X_{0n} \rangle^T$$

and a point satisfying Equation (**) as equality, we can perceive the solution space of

the equation:

$$a_1 X_1 + a_2 X_2 + \dots + a_n X_n = b$$

Hyperplane

Is solution space of an *n*-var linear equation of the general form:

$$a_1 X_1 + a_2 X_2 + \dots + a_n X_n = b$$

Isoprofit lines

are lines for the objective function that are drawn to locate the optimal value when using the geometric approach.

Linear programming problem

is a set of (linear) inequalities (with a solution set S) and a (linear) function (often cost or profit) whose value (within S) is to be maximized or minimized.

Marginal analysis

is concerned with the effects of small perturbations, maybe measurable by derivatives.

Nonbasic Variables

A variable not in the basic solution (value = 0).



Objective Function

The function that is either being minimized or maximized. For example, it may represent the cost that you are trying to minimize.

Operation research

sometimes known as management science, refers to the constant scientific monitoring of an organisation's ongoing activities, focussing on decision and control problems that affect daily operations of the organisation. It involves the representation of real world situations by mathematical models, together with the use of quantitative methods (algorithms) for solving such models with a view to optimizing.

Optimal Solution (vector)

A vector x which is both feasible (satisfying the constraints) and optimal (obtaining the largest or smallest objective value).

Optimal Solution

is the maximum or minimum feasible solution which is obtained at final stage of the simplex method algorithm process.

Parametric analysis

is concerned with larger changes in parameter values that affect the data in the mathematical program, such as a cost coefficient or resource limit.

Phase I & Phase II

Phase I of a mathematical program is finding a **feasible** solution, and Phase II is entered with a feasible solution to find an **optimal** solution.

Pivot. This is the algebra associated with an iteration of Gauss-Jordan elimination, using the *forward transformation*. (refer to illustration below)

$$\left[\begin{array}{cccccc|c} 2 & 1 & 1 & 1 & 0 & 0 & 14 \\ 4 & 2 & 3 & 0 & 1 & 0 & 28 \\ 2 & 5 & 5 & 0 & 0 & 1 & 30 \\ \hline -1 & -2 & +1 & 0 & 0 & 0 & 0 \end{array} \right]$$

↑ Pivot column
 ↙ Pivot
 ↖ Pivot row

Polytope

is solution space (feasible region) of an n -var linear programming is geometrically defined by the intersection of a number of half-spaces and/or hyperplanes equal to the linear programming constraints, including the *sign restrictions*.

Polyhedron

is a bounded polytope



Pricing

This is a tactic in the **simplex method**, by which each variable is evaluated for its potential to improve the value of the objective function.

Primal degenerate

is a pivot where the associated basic solution (x) does not change (i.e., the non-basic variable enters the basis, but its level remains at the same bound value, in which case no basic variable changes level).

Sensitivity analysis

The concern with how the solution changes if some changes are made in either the data or in some of the solution values (by fixing their value).

Simplex method

An algorithm invented to solve a linear program by progressing from one extreme point of the feasible polyhedron to an adjacent one. The method is an *algorithm strategy*, where some of the tactics include *pricing* and *pivot selection*.

Simplex

(pl. *simplices*). $\{x \text{ in } \mathbb{R}: \text{Sum}\{x_j\} = 1\}$. For $n=1$, this is a point ($x=1$). For $n=2$, this is a line segment, joining points $(1,0)$ and $(0,1)$. For $n=3$, this is a triangle, joining the vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. This is sometimes called an *n-simplex*, denoted by S_n (note its dimension is $n-1$). The *open simplex* excludes the axes: $\{x \text{ in } S_n: x > 0\}$.

Slack variable

In an inequality constraint of the form $g(x) \leq b$, the slack is $b-g(x)$, which is designated by the slack variable, s . Then, the original constraint is equivalent to the defining equation, $g(x) + s = b$, plus $s \geq 0$.

Slack Variable

A variable added to the problem to eliminate less-than constraints.

Standard Maximizing Problem

is a linear programming problem which satisfies all of the following 4 conditions:

- 1) The objective function is to be maximized.
- 2) All inequalities are of the \leq type.
- 3) All right hand constants are non-negative.
- 4) All variables are non-negative.

[A Non-Standard Problem

is simply a problem which is not standard, and hence fails to satisfy at least one of [1] through [4] above.]

Surplus Variable

A variable added to the problem to eliminate greater-than constraints.

**Tableau**

(pl. *tableaux*). A detached coefficient form of a system of equations, which can change from $x + Ay = b$ to $x' + A'y' = b'$. The primes denote changes caused by multiplying the first equation system by the basis inverse (a sequence of pivots in the simplex method).

Unbounded Solution

For some linear programs it is possible to make the objective arbitrarily small (without bound). Such a linear programming is said to have an unbounded solution.



XII. Compiled List of Compulsory Readings

Linear Programming: Foundations and Extensions by Robert J. Vanderbei

www.princeton.edu/~rvdb/LPbook/onlinebook.pdf

All learners are required to read this book whenever he/she is starting a new unit or activity. All other links are to help the learner to understand more about Linear Programming problems



XIII. Compiled List of (Optional) Multimedia Resources

Reading 1: Wolfram MathWorld (*visited 03.11.06*)

Complete reference : <http://mathworld.wolfram.com>

Abstract : Wolfram MathWorld is a specialised on-line mathematical encyclopedia.

Rationale : It provides the most detailed references to any mathematical topic. Students should start by using the search facility for the module title. This will find a major article. At any point students should search for key words that they need to understand. The entry should be studied carefully and thoroughly.

Reading 2 : Wikipedia (*visited 03.11.06*)

Complete reference : <http://en.wikipedia.org/wiki>

Abstract : Wikipedia is an on-line encyclopedia. It is written by its own readers. It is extremely up-to-date as entries are continually revised. Also, it has proved to be extremely accurate. The mathematics entries are very detailed.

Rationale : Students should use wikipedia in the same way as MathWorld. However, the entries may be shorter and a little easier to use in the first instance. They will, however, not be so detailed.

Reading 3 : MacTutor History of Mathematics (*visited 03.11.06*)

Complete reference : <http://www-history.mcs.standrews.ac.uk/Indexes>

Abstract : The MacTutor Archive is the most comprehensive history of mathematics on the internet. The resources are organised by historical characters and by historical themes.

Rationale : Students should search the MacTutor archive for key words in the topics they are studying (or by the module title itself). It is important to get an overview of where the mathematics being studied fits in to the history of mathematics. When the student completes the course and is teaching high school mathematics, the characters in the history of mathematics will bring the subject to life for their students. Particularly, the role of women in the history of mathematics should be studied to help students understand the difficulties women have faced while still making an important contribution.. Equally, the role of the African continent should be studied to share with students in schools: notably the earliest number counting devices (e.g. the Ishango bone) and the role of Egyptian mathematics should be studied.



The following sites enable you to practice the Linear Programming pivoting without having to do the arithmetic!

1. Pivot Tool: <http://www.sor.princeton.edu/~rvdb/JAVA/pivot/advanced.html>
2. TUTOR: <http://www.tutor.ms.unimelb.edu.au/>
3. The Simplex Place: <http://www.ifors.org/tutorial>
4. Simplex Machine: <http://www.ms.unimelb.edu.au/~moshe/lp/simplex7.html>
5. linear programming-Explorer: http://www.maths.ed.ac.uk/linear_programming-Explorer
6. Taha, H. (2003, 7th ed). TORA Solver: Operation Research: An Introduction



XIV. Compiled List of Useful Links

Linear Programming Formulation

<http://people.brunel.ac.uk/~mastjjb/jeb/or/lpmore.html>

Linear Programming Formulation

<http://people.brunel.ac.uk/~mastjjb/jeb/or/lpmore.html>

Last visited: 15-02-07

Linear Programming Formulation

<http://home.ubalt.edu/ntsbarsh/opre640a/partVIII.htm>

Last visited: 15-02-07

An Introduction to Linear Programming and the Simplex Algorithm by Spyros Reveliotis http://www2.isye.gatech.edu/~spyros/linear_programming/linear_programming.html

Last visited: 14-02-07

(1) Linear Programming

<http://home.ubalt.edu/ntsbarsh/opre640a/rpcotdp#rpcotdp>

Last visited: 21-02-07

(2) Linear Programming Formulation

www.people.brunel.ac.uk/~mastjjb/jeb/lp.html

Linear Programming: A Geometric Approach page 171-176

<http://www.wiley.com/college/sc/sullivan/CH03.pdf>

Optimality Conditions for Constrained Optimization

http://ocw.mit.edu/NR/rdonlyres/Sloan-school-of-Management/15-084JSpring2004/7240EF84-B20D-419F-B1C0-2DAF3277F5C4/0/lec6_constr_opt.pdf

Last visited: 16-02-07



Mathematical approach of checking conditions of optimality

Optimality Conditions

<http://www.math.mtu.edu/%7Emsgocken/ma5630spring2003/lectures/lag1/lag1/node1.html>

Last visited: 16-02-07

Alternate Optimal Solutions, Degeneracy, Unboundedness, Infeasibility

<http://mat.gsia.cmu.edu/QUANT/notes/node63.html#SECTION0083000000000000000000>

Degeneracy and unboundedness are well explained with aid of simple examples.

The Fundamental Theorem of Linear Programming

http://www.it.uu.se/edu/course/homepage/opt1/ht06/Lectures/fundamentat_thm-2up.pdf

<http://enr.smu.edu/~barr/ip/ch1/node7.html>

<http://www.maths.abdn.ac.uk/~igc/tch/mx3503/notes/node67.htm>

These links provides theorems of linear programming and the relevant proofs

Simplex Method - Big M

http://www.math.uwo.ca/~heinicke/courses/236_03/bigM.pdf

Last visited: 15-02-07

Simplex Method – Big M

<http://www.computing.dcu.ie/~lkillen/teach/CA427Simplexbigmexample.pdf>

Last visited: 15-02-07

The algorithm of the B-M method is exemplified

Dual Problem: Construction and Its Meaning

<http://home.ubalt.edu/ntsbarsh/opre640a/rpcotdp#rpcotdp>

This link takes the student step by step in the construction of the dual problem. The association between the primal and dual problem is well illustrated with good examples.

Sensitivity Analysis for Linear Programming

<http://mat.gsia.cmu.edu/QUANT/notes/node64.html>

Last visited: 20-02-07

This link gives a good explanation on “*Why sensitivity analysis?*”.

Deterministic Modeling: Linear Optimization with Applications

<http://www.mirror-service.org/sites/home.ubalt.edu/ntsbarsh/Business-stat/opre/partVIII.htm#online>

This link explains the flow of the linear programming problem from formulation down to sensitivity analysis. One problem is taken from formulation stage to sensitivity analysis stage. Students are encouraged to visit this site whenever they have problems of understanding some of the linear programming technical terms and processes.



Duality in Linear Programming

<http://www.web.mit.edu/15.053/www/AMP-Chapter-04.pdf>

Explains the duality theorem and duality properties

Basic Solutions

<http://engr.smu.edu/~barr/ip/ch1/node6.html>

Linear Programming: Geometric Approach

www.math.tamu.edu/~janice.epstein/141/notes/Ch3.pdf

Last visited date 16-02-07



XV. Synthesis of the Module

In this module you have been introduced to an area of mathematics called Linear Programming that plays a central role in a wider field called Operations Research. Operations Research deals with situations where one is interested in optimising (minimising or maximising) a specific quantity that is often framed as a function of several influencing factors (variables) with an overall goal of making decisions related to the behaviour of the situation of interest. Linear programming offers a technique for resolving a class of problematic situations of that nature.

To develop a feel for mathematics related to Linear Programming, you have learned the following main ideas:

- Formulation of a linear programming problem. [Identify constraints of the situation and the objective function to be optimised, then construct a system of inequality relations involving them].
- Geometrical interpretation of a solution of a linear programming problem. [The optimal solution is located at one corners where boundary lines of the feasible region intersect, or along one of the boundary lines]

Optimality conditions for the objective function of a linear programming problem. [In the step-by-step procedure for eliminating basic variables in the initial tableau constructed from the objective function and its constraints, an indication that an optimal value has been reached is that for a maximization problem, the coefficients of the basic feasible variables in the objective function should be negative, meaning that the variables can no longer contribute to increasing the value of the objective function. Similarly for a minimization problem]

- Algebraic interpretation of the solution to a linear programming problem. [This is the set of remaining nonbasic feasible variables after all basic feasible variables have been eliminated in the step by step procedure from the initial tableau matrix constructed from the objective function and its constraints.]
- Sensitivity analysis. [a solution is stable if small changes introduced into its constraint variables do not produce large changes in the value of the objective function associated with the solution].

In short, the module looked at formulation of the linear programming problem and its solution using the geometric approach (graphical method) and the algebraic method (simplex method). Stability of the solutions was checked using sensitivity analysis.

The authors of the module believe that the learner who has successfully developed a fairly good understanding of the main concepts and mastery of the associated skills (at least as indicated by the assessment activities in-built in the module) will not only be able to teach the introductory levels of linear programming concepts at secondary school level, but will also be in a position to proceed further within the field of Operations Research.



XVI. Formative and Summative Evaluation

In this module evaluation of student learning comes in 2 components: the formative (continuous or coursework) and the summative (cumulative) components. The formative component is prompted by the various exercises appearing in all the learning activities, with the evaluating process being executed by the student him/herself, that is to say, they are self-evaluation learning tasks or exercises for student's direct benefit. The intent of that evaluation is to give the student a sense of the progress the student is making in learning the material as he/she moves through the module. That evaluation can also be recorded by the student [see provision for that on the student records Excel spreadsheet in section 18 below.] Recording his/her progress is important for him/her to build his/her learning profile for the material in the module and keep track of the variations in levels of understanding or difficulty as the student moves from learning activity to learning activity. This formative self-evaluation is best done using qualitative categories such as poor, satisfactory, good, and very good. It is expected that the student will conduct the self-evaluation process in as honest a manner as possible because dishonest self-assessing will only serve to undermine the student to his/her detriment.

The second component of the evaluation, the summative evaluation, is accomplished through a formal written test consisting of 10 items that can be completed in 4 hours on the average. The test will be administered in person by the course program providers at a designated learning centre, where the students will also sit for the test in person. Students will require some graph paper and graph plotting accessories, and a standard scientific or programmable calculator, whichever they are comfortable with using. The test will be marked or scored and evaluated by the course program providers and the student's performance in the test will be taken to represent the amount of cumulative learning and understanding of the material in the module that the student has attained. That assessed performance will in turn provide the final course mark or grade obtained by the student for this course or module on linear programming.

A genuine attempt has been made to select or design test items that assess the extent the student has grasped the essential features and characteristics of the general linear programming problem described in Unit 1 box of the Flow of Learning chart in the earlier overview section [see section 7], and developed associated competencies described in the Unit 2 box of the same chart. A description of the test items and the related learning objective each item is intended to assess, and the list of the actual test items now follow below.

The answers to the test items are also supplied at the end of the test item listing.

Acknowledgement: [Some of the questions have been adapted from selected exercises in Wagner, H. M. (1975, 2nd ed). Principles of Principles of Operations Research. Englewood Cliffs, New Jersey: Prentice-Hall].



Question 1 tests the student's understanding of a linear programming problem situation through the student's ability to formulate the linear programming problem properly (i.e. writing the problem in mathematical language).

Questions 2 and 3 test the student's capability to write the problem in mathematical language and solve it using the graphical (geometric) method. Graph paper is relevant for accuracy of diagrams.

Question 4 tests the student's capability to solve the linear programming problem using the Simplex Method.

Questions 5 and 6 test the student's capability to solve the linear programming problem using the Big- M Method followed by the Simplex Method. The student should know when it is appropriate to use the Big- M method.

Question 7 checks if the student can identify the primal and dual forms in a linear programming situation and formulate the dual of a given primary linear programming problem.

Question 8 tests the student's capability to investigate the feasibility and boundedness characteristics of a linear programming problem.

Question 9 tests the student's capability to solve the linear programming problem using the Dual Simplex method.

Question 10 tests the student's capability to solve a linear programming problem using the Simplex Method and then carry out a sensitivity analysis on the solution obtained.



Question 1

The Mufoya Bicycle Company produces two kind of bicycles by hand: Mountain Bike and Street Racers. Mufoya wishes to find out the rate at which each type of bicycle should be produced in order to maximize the profits on the sales of the bikes. Mufoya assumes that the company can sell all the bicycles produced. The physical data on the production process is available from the company engineer. A different team produces each kind of bicycle, each team has a different maximum production rate: 2 mountain bikes per day and 3 racers per day respectively. Producing a bicycle of either type requires the same amount of time on the metal finishing machine (a production bottle-neck), and this machine can process at most a total of 4 bicycles per day, of either type. The company accountant estimates that mountain bikes are currently generating a profit of \$15 per bicycle and racers are giving a profit of about \$10 per bicycle. Formulate the Linear Programming Problem.

Questions 2 and 3

A manufacturer of electronic instruments produces two types of timers: a standard and a precision model with net profits of \$2 and \$3 respectively. They are similar in design, each taking about the same amount of time to assemble. Let us suppose that the manufacturer wishes to maximize his net profit each day subject to the availability of resources and marketing considerations. Let's consider that his work force can produce no more than 50 instruments per day. Furthermore, suppose there are 4 main components in short supply (a,b,c and d), and that they are used in different quantities for the types of timers as shown below.

Component	The Stock Constraints		
	Stock	Standard (x)	Precision (y)
a	220	4	2
b	160	2	4
c	370	2	10
d	300	5	6

That is to say each of the x standard timers uses 4 of component a, and each of the y precision ones uses 2, and the factory cannot use more than what is in stock.

[A] Formulate the Linear Programming Problem.

[B] (i) Find the feasible region of the Linear Programming problem using the geometrical method.

(ii) Find the value of the maximum profit.

Note: You need GRAPH PAPER for this question.

**Question 4**

Use the Simplex method to solve the following linear programming problem

$$\text{Maximize } 4x_1 + 5x_2 + 9x_3 + 11x_4$$

Subject to

$$x_1 + x_2 + x_3 + x_4 \leq 15$$

$$7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 120$$

$$3x_1 + 5x_2 + 10x_3 + 15x_4 \leq 100$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Question 5

(a) Explain fully when the Big M method is used.

(b) Maximize $3x_1 + 4x_2$

Subject to

$$2x_1 + x_2 \leq 600$$

$$x_1 + x_2 \leq 225$$

$$5x_1 + 4x_2 \leq 1000$$

$$x_1 + 2x_2 \geq 150$$

$$x_1, x_2 \geq 0$$

Question 6

Maximize $-3x_1 - 2x_2$

Subject to

$$x_1 + x_2 = 10$$

$$x_1 \geq 4$$

$$x_1 \geq 0, x_2 \geq 0$$



Question 7

- (a) Explain what is meant by the primal and dual of a linear programming problem.
- (b) Formulate the dual of the linear programming problem below

$$\text{Maximize } 4x_1 + 5x_2 + 9x_3 + 11x_4$$

Subject to

$$x_1 + x_2 + x_3 + x_4 \leq 15$$

$$7x_1 + 5x_2 + 3x_3 + 2x_4 \leq 120$$

$$3x_1 + 5x_2 + 10x_3 + 15x_4 \leq 100$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Question 8

Investigate the feasibility of the problem below

a) Maximize $5x_1 + 4x_2$

subject to

$$x_1 + x_2 \leq 2$$

$$-2x_1 - 2x_2 \leq -9$$

$$x_1, x_2 \leq 0.$$

- b) Determine the boundedness of the problem below

Maximize $x_1 - 4x_2$

subject to

$$-2x_1 + x_2 \leq -1$$

$$-x_1 - 2x_2 \leq -2$$

$$x_1, x_2 \geq 0.$$

Question 9

Minimize $2x_1 + x_3$

Subject to

$$x_1 + x_2 - x_3 \geq 5$$

$$x_1 - 2x_2 + 4x_3 \geq 8$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

**Question 10**

Consider the Linear Programming problem below

$$\text{Maximize } z = 15x_1 + 10x_2$$

Subject to

$$x_1 \leq 2$$

$$x_2 \leq 3$$

$$x_1 + x_2 \leq 4$$

The optimal solution is $x_1 = 2$, $x_2 = 2$, $z = 50$

Suppose that the third constraint is changed to $x_1 + x_2 \leq 3$. That is to say the new problem becomes:

$$\text{Maximize } z = 15x_1 + 10x_2$$

Subject to

$$x_1 \leq 2$$

$$x_2 \leq 3$$

$$x_1 + x_2 \leq 3$$

Investigate whether or not the solution to the Linear Programming problem is sensitive to the change in the constraint.



Answers to questions

Question 1

The first step is to identify the variables. These are the values you can set or otherwise control. The Mufoya variables are the production rates of the mountain bikes, x_1 and racers, x_2 .

The objective function is as follows:

Maximize daily profit, i.e., maximize $z = 15x_1 + 10x_2$ (in \$ per day)

The constraints are as follows:

Mountain bikes production limit $x_1 \leq 2$ (bikes per day)

Racers production limit $x_2 \leq 3$ (bikes per day)

Metal finishing machine production limit $x_1 + x_2 \leq 4$ (bikes per day)

Questions 2 and 3

[A] Maximize $2x + 3y$

subject to:

$$x + y \leq 50$$

$$4x + 2y \leq 220$$

$$2x + 4y \leq 160$$

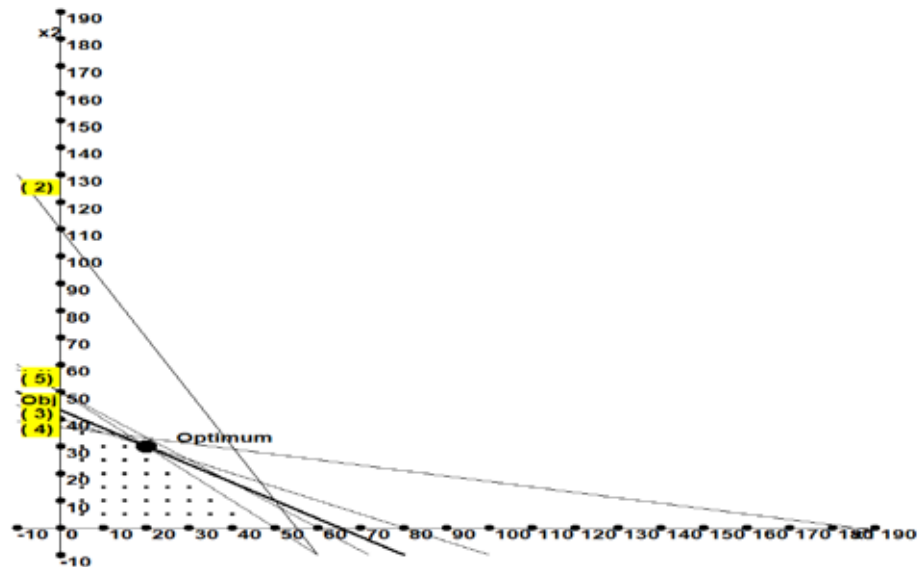
$$2x + 10y \leq 370$$

$$5x + 6y \leq 300$$

$$x, y \geq 0$$



[B] Sketch:



Summary of optimal solution

Objective Value = 130

$$x = 20.00$$

$$y = 30.00$$

Question 4

Let x_0 be the value of the objective function add the slack variables x_5, x_6, x_7 to your constraints so that you have equality.

Then write the system as

$$x_0 - 4x_1 - 5x_2 - 9x_3 - 11x_4 = 0$$

$$x_1 + x_2 + x_3 + x_4 + x_5 = 15$$

$$7x_1 + 5x_2 + 3x_3 + 2x_4 + x_6 = 120$$

$$3x_1 + 5x_2 + 10x_3 + 15x_4 + x_7 = 100$$



After going through four iterations of the Simplex Method you get to a stage where the non-basic variables x_2, x_4, x_5 or x_7 can only have value zero.

That is to say we have:

$$x_0 = \frac{695}{7} - \frac{3}{7}x_2 - \frac{11}{7}x_4 - \frac{13}{7}x_5 - \frac{5}{7}x_7$$

Question 5

Standard form:

$$\text{Maximize } 3x_1 + 4x_2$$

Subject to

$$2x_1 + 3x_2 + s_1 = 600$$

$$x_1 + x_2 + s_2 = 225$$

$$5x_1 + 4x_2 + s_3 = 1000$$

$$x_1 + 2x_2 - s_4 = 150$$

$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

Not in canonical form because there is no basic variable in the fourth equation. Therefore we add an artificial variable to that equation (r_1) and give it a large **negative** coefficient in the objective function, to penalize it:

$$\text{Maximize } 3x_1 + 4x_2$$

Subject to

$$2x_1 + x_2 + s_1 = 600$$

$$x_1 + x_2 + s_2 = 225$$

$$5x_1 + 4x_2 + s_3 = 1000$$

$$x_1 + 2x_2 - s_4 + r_1 = 150$$

$$x_1, x_2, s_1, s_2, s_3, s_4, r_1 \geq 0$$



x_1	x_2	s_4	s_1	s_2	s_3	r_1	B
Z	-3	-4	0	0	0	0	+M
s_1	2	3	0	1	0	0	600
s_2	1	1	0	0	1	0	225
s_3	5	4	0	0	0	1	1000
r_1	1	2	-1	0	0	0	150

Not in Canonical form because of +M entry on Z row for one basic variable (r_1).
 Pivot to replace +M on Z row by zero - Z row - M* r_1 row:

x_1	x_2	s_4	s_1	s_2	s_3	r_1	B
Z	(-3 -M)	(-4 -2 M)	0	0	0	0	-150M
s_1	2	3	0	1	0	0	600
s_2	1	1	0	0	1	0	225
s_3	5	4	0	0	0	1	1000
r_1	1	2	-1	0	0	0	150

x_1	x_2	s_4	s_1	s_2	s_3	r_1	b
Z	-1	0	-2	0	0	0	M 800
s_1	$\frac{1}{2}$	0	$\frac{3}{2}$	1	0	0	$-\frac{3}{2}$ 375
s_2	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	0	$-\frac{1}{2}$ 150
s_3	3	0	2	0	0	1	-2 700
x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$ 75

x_1	x_2	s_4	s_1	s_2	s_3	r_1	b
Z	-1/3	0	0	4/3	0	0	M 800
s_4	1/3	0	1	2/3	0	0	-1 250
s_2	1/3	0	0	-1/3	1	0	0 25
s_3	7/3	0	0	-4/3	0	1	0 200
x_2	2/3	1	0	1/3	0	0	0 200



x_1	x_2	s_4	s_1	s_2	s_3	r_1	b	
Z	0	0	0	1	1	0	M	825
s_4	0	0	1	1	-1	0	-1	225
x_1	1	0	0	-1	3	0	0	75
s_3	0	0	0	1	-7	1	0	25
x_2	0	1	0	1	-2	0	0	250

Optimal tableau: Solution: $x_1^* = 75$ $x_2^* = 150$ $Z^* = 825$

Question 6

Then after adding a surplus variable x_3 in the inequality above, you can write the model as

$$x_0 + 3x_1 + 2x_2 = 0$$

$$x_1 + x_2 = 10$$

$$x_1 - x_3 = 4$$

Next, introduce artificial variables y_1 and y_2 , and let $M=10$, giving

$$x_0 + 3x_1 + 2x_2 + 10y_1 + 10y_2 = 0$$

$$x_1 + x_2 + y_1 = 10$$

$$x_1 - x_3 + y_2 = 4$$

To initiate the simplex algorithm, you have to subtract ($M=10$) times row 2 and ($M=10$) times row 3 from row 1 to eliminate y_1 and y_2 :

$$x_0 - 17x_1 - 8x_2 + 10x_3 = -140$$

$$x_1 + x_2 + y_1 = 10$$

$$x_1 - x_3 + y_2 = 4$$

The optimal solution $x_1 = 4$ and $x_2 = 6$



Question 7

Minimize $15y_1 + 120y_2 + 100y_3$

Subject to.

$$y_1 + 7y_2 + 3y_3 \geq 4$$

$$y_1 + 5y_2 + 5y_3 \geq 5$$

$$y_1 + 3y_2 + 10y_3 \geq 9$$

$$y_1 + 2y_2 + 15y_3 \geq 11$$

$$y_1, y_2, y_3 \geq 0$$

Question 8

- a) The second constraint implies that, $x_1 + x_2 \geq 4.5$, which contradicts the first constraint. If a problem has no feasible solution, then the problem itself is called *infeasible*.
- b) Here, we could set x_2 to zero and let x_1 be arbitrarily large. As long as x_1 is greater than 2, the solution will be feasible, and as it gets large the objective function does too. Hence, the problem is unbounded.

Question 9

SIMPLEX TABLEAUS -- (Dual Simplex Method)

Title: Question 9

Iteration 1	x	y	z			
Basic	x1	x2	x3	Sx4	Sx5	Solution
z (min)	-2.00	0.00	-1.00	0.00	0.00	0.00
Sx4	-1.00	-1.00	1.00	1.00	0.00	-5.00
Sx5	-1.00	2.00	-4.00	0.00	1.00	-8.00
Lower Bound	0.00	0.00	0.00			
Upper Bound	infinity	infinity	infinity			
Unrestr'd (y/n)?	n	n	n			

Iteration 2	x	y	z			
Basic	x1	x2	x3	Sx4	Sx5	Solution
z (min)	-1.75	-0.50	0.00	0.00	-0.25	2.00
Sx4	-1.25	-0.50	0.00	1.00	0.25	-7.00
x3	0.25	-0.50	1.00	0.00	-0.25	2.00
Lower Bound	0.00	0.00	0.00			
Upper Bound	infinity	infinity	infinity			
Unrestr'd (y/n)?	n	n	n			

Iteration 3	x	y	z			
Basic	x1	x2	x3	Sx4	Sx5	Solution
z (min)	-0.50	0.00	0.00	-1.00	-0.50	9.00
x2	2.50	1.00	0.00	-2.00	-0.50	14.00
x3	1.50	0.00	1.00	-1.00	-0.50	9.00
Lower Bound	0.00	0.00	0.00			
Upper Bound	infinity	infinity	infinity			
Unrestr'd (y/n)?	n	n	n			

**Question 10**

You can solve the problem geometrically to find the optimal solution is $x_1 = 2$, $x_2 = 1$, $z = 40$. Since z and x_2 changed when an original coefficient was changed, then we say that the L.P. is sensitive.



XVII. References

Vanderbei R. J. (year). Linear Programming: Foundations and Extensions

www.princeton.edu/~rvdb/LPbook/onlinebook.pdf

All learners are required to read this book whenever he/she is starting a new unit or activity. All other links are to help the learner to understand more about Linear Programming problems

A gentle approach to linear programming

<http://www.sce.carleton.ca/faculty/chinneck/po/Chapter1.pdf>

J E Beasley , OR-Notes

<http://people.brunel.ac.uk/~mastjjb/jeb/or/twomines>

OR-NOTES provides a strong and easy to understand approach to the formulation of the linear programming problems. The examples are well explained. Students are encouraged to go through some of the worked examples to strengthen their problem linear programming problem formulation techniques.

Varaiya, P. Lecture notes on Optimization

http://robotics.eecs.berkeley.edu/~varaiya/papers_ps.dir/NOO.pdf

These notes go hand in hand with the main textbook

Management Science: Linear Programming Notes

<http://www.strathcona.bham.ac.uk/Pdfs1%20management%20Course%20Ye%202/LINEAR%20PROGRAMM%20NOTES.PDF>

These notes outline the problem formulation, graphical solution of linear programming problems and the sensitivity analysis

Spyros,R. An Introduction to Linear Programming and the Simplex Algorithm

http://www2.isye.gatech.edu/~spyros/linear_programming/linear_programming.html

McCarl, B. and Spreen, T. H., 2002. DUALITY IN LINEAR PROGRAMMING.

Baker, S. L. (2006) Linear Programming I: Maximization

Linear Programming: A Geometric Approach

<http://www.wiley.com/college/sc/sullivan/CH03.pdf>

Basic Solutions

<http://enr.smu.edu/~barr/ip/ch1/node6.html>

Arsham (2007)

<http://home.ubalt.edu/ntsbarsh/opre640a/partVIII.htm>



The following references were also used by the authors

- Wagner, H. M. (1975, 2nd ed). *Principles of Principles of Operations Research*. Englewood Cliffs, New Jersey: Prentice-Hall.
- Spencer, A. J. M., Parker, D. F., Berry, D. S., England, A. H., Faulkner, T. R., Green, W. A., Holden, J. T., Middleton, D., & Rogers, T. G. (1977, reprint 1981). *Engineering Mathematics*. London: Van Nostrand Reinhold Co Ltd.
- Koshy, T. (1979). *Finite Mathematics and Calculus (with application)*. Santa Monica, CA: Goodyear Publishing Co. Inc
- Taha, H. (2003, 7th ed). *TORA Solver: Operation Research: An Introduction*

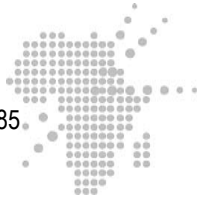


XVIII. File Structure

University Name.....

	Name	Reg Number	Gender	Pre-Assessment	Activity 1	Activity 2	Activity 3	Activity 4	Activity 5	Activity Ave	Exam	Grade
1												
2												
3												
4												
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Lecturer..... Signature..... Date.....
 Coordinator..... Signature..... Date.....
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XIX. Main Author of the Module

This module was developed by David K J Mtetwa, BSc, MSc, MEd, PhD, Grad Cert Edu; in collaboration with Admire Kurira, BSc, MSc; and Blessing Mufoya, BSc (Hons), MSc, who are all mathematics teacher educators based in the Department of Science and Mathematics Education at the University of Zimbabwe.

Dr Mtetwa was born and grew up in Eastern Zimbabwe, where he completed his primary and secondary education. After obtaining his undergraduate degree majoring in mathematics and physics at the University of Lesotho, Dr Mtetwa enrolled for a masters degree in mathematics at a Canadian University. This was followed by a number of teaching stints at a number of high schools and university colleges in Lesotho, Swaziland, and Zimbabwe. His teaching certification training was completed at the University of Zimbabwe in 1984, following which he crossed the Atlantic again, but his time to the United States to undertake Masters and PhD degrees in mathematics education at the state universities of New York and Virginia, that were completed in 1991. Dr Mtetwa has since then been teaching and developing mathematics education courses for diploma and degree levels at the University of Zimbabwe. He has supervised and graduated many post-graduate students in the area of mathematics education. Dr Mtetwa is an active member of various professional associations, including the influential Southern African Association for Research in Mathematics, Science, and Technology Education (SAARMSTE) and African Commission on Mathematics Education (AFRCME).